Double coverings of toric singularities

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Abstract

We show that all simple elliptic singularities, two dimensional cusp singularities of restricted type and certain higher dimensional singularities, are double coverings of toric singularities. As an application, we give a process of determining defining equations and calculating plurigenera of such singularities.

Let $N = \mathbf{Z}^r$ for an integer r greater than 1. Let σ be an r-dimensional rational polyhedral convex cone in N and let Y be the corresponding toric singularity. Let $M = \mathrm{Hom}(N, \mathbf{Z})$ and let $\sigma^\vee$ be the dual cone of σ. For an element P in $\mathbf{C}[\sigma^\vee \cap M]$ the normalization W of the subvariety of $Y \times \mathbf{C}$ defined by $w^2 - P = 0$, is a double covering of Y. We give in Section 1, a resolution of singularities of W and in Section 2, a process to determine defining equations of W for P satisfying a certain condition. In Section 3, we show that all simple elliptic singularities and 'symmetric' two dimensional cusp singularities are obtained as above W, and determine defining equations of some examples, where 'symmetric' means that the weighted dual graph of the exceptional set of the minimal resolution has an automorphism of order 2 with fixed points. In Section 4, we give a necessary and sufficient condition for W to become $\mathbf{Q}$-Gorenstein or log canonical. In Section 5, in the case that W is an isolated singularity, we give a process of calculating plurigenera and κ_δ defined in [2] and [5]. We also give examples of singularities V with $1 \le \kappa_\delta \le \dim V - 2$. We use notations in [3], freely. However, we denote by $\mathbf{e}^m$ instead of $\mathbf{e}(m)$, the character of an element m in M.

1 Toric singularities and their double coverings

Let N and M be as in Introduction, and let $\langle\ ,\ \rangle\colon M \times N \to \mathbf{Z}$ be the natural bilinear map. Let σ be an r-dimensional rational polyhedral convex cone in N. Then the dual cone $\sigma^\vee := \{x \in M_{\mathbf{R}} \mid \langle x, y\rangle \ge 0 \text{ for } y \in \sigma\}$ of σ is also an r-dimensional rational polyhedral convex cone. Let Σ be a non-singular fan in N with $|\Sigma|$ $(= \bigcup_{\tau\in\Sigma} \tau) = \sigma$ and let $\Sigma(i) = \{\tau \in \Sigma \mid \dim\tau = i\}$. Let $Y = T_N\mathrm{emb}(\{\text{faces of } \sigma\})$ $(= \mathrm{Spec}\mathbf{C}[\sigma^\vee \cap \mathrm{M}])$, $y_0 = \mathrm{orb}(\sigma)$, $\widetilde{Y} = T_N\mathrm{emb}(\Sigma)$ and let $\pi : \widetilde{Y} \to Y$ be the natural projection. Then $\widetilde{Y}$ is non-singular, $\widetilde{Y} \setminus T_N = \sum_{\tau\in\Sigma(1)} V(\tau)$ and the exceptional set of π is $E := \bigsqcup_{\tau\in\Sigma_{\mathrm{in}}} \mathrm{orb}(\tau)$, where $V(\tau) = \overline{\mathrm{orb}(\tau)}$, $\Sigma_{\mathrm{in}} = \{\tau \in \Sigma \mid \tau \not\prec \sigma\}$. For a cone τ in Σ, we denote by $\hat\tau$ the minimal face of σ containing τ, and by $\pi(\tau)$ the restriction of π to $V(\tau)$. We see by [3, Theorem 1.15] that fibers of $\pi(\tau) : V(\tau) \to V(\hat\tau)$ are compact. In particular, $V(\tau)$ is compact, if and only if $\hat\tau = \sigma$. In the case that $\dim\hat\tau = \dim\tau + 1$, generic fibers of $\pi(\tau)$ are biholomorphic to $\mathbf{P}^1$. Let $\Sigma_{\mathrm{c}} = \{\tau \in \Sigma \mid \hat\tau = \sigma\}$ and let $E_{\mathrm{c}} = \bigsqcup_{\tau\in\Sigma_{\mathrm{c}}} \mathrm{orb}(\tau)$. Then $\pi^{-1}(y_0) = E_{\mathrm{c}}$.

Let K be a finite subset of $M \cap \sigma^\vee \setminus \{0\}$ with $|K| \ge 2$ and let $\mathrm{Min}(K, u) = \min\{\langle m, u\rangle \mid m \in K\}$ for an element u in N. Let u_τ be the primitive element in N spaning τ for a 1-dimensional cone τ in Σ. Then for an element m in M, the character $\mathbf{e}^m$ has zero of order $\langle m, u_\tau\rangle$ along $V(\tau)$. Let $P = \sum_{m\in K} c_m\mathbf{e}^m$ for non-zero complex numbers c_m. Then there exists a divisor B in $\widetilde{Y}$ containing no common divisors in $\widetilde{Y} \setminus T_N$ such that

$$(P) = B + \sum_{\tau\in\Sigma(1)} \mathrm{Min}(K, u_\tau)V(\tau).$$

Let $\tau \neq \{0\}$ be a cone in Σ and let μ be an r-dimensional cone in Σ containing τ. Then there exists a basis $\{u_1, u_2, \ldots, u_r\}$ of N such that $\mu = \mathbf{R}_{\geq 0}u_1 + \cdots + \mathbf{R}_{\geq 0}u_r$, $\tau = \mathbf{R}_{\geq 0}u_1 + \cdots + \mathbf{R}_{\geq 0}u_s$ ($s = \dim \tau$). Let $\{v_1, v_2, \ldots, v_r\}$ be the basis of M dual to $\{u_1, u_2, \ldots, u_r\}$ and let $z_i = \mathbf{e}^{v_i}$. Then z_i vanishes along $V(\tau)$ for $1 \leq i \leq s$. Since $\mathbf{e}^m = z_1^{\langle m,u_1\rangle} \cdots z_r^{\langle m,u_r\rangle}$ for any element m in M,

$$P = z_1^{a_1} \cdots z_s^{a_s} \left(\sum_{m \in K(\tau)} c_m z_{s+1}^{\langle m,u_{s+1}\rangle} \cdots z_r^{\langle m,u_r\rangle} \right. \\ \left. + \sum_{m \in K \setminus K(\tau)} c_m z_1^{\langle m,u_1\rangle - a_1} \cdots z_s^{\langle m,u_s\rangle - a_s} z_{s+1}^{\langle m,u_{s+1}\rangle} \cdots z_r^{\langle m,u_r\rangle} \right),$$

where $a_i = \mathrm{Min}(K, u_i)$ and $K(\tau) = \{m \in K \mid \langle m, u_i\rangle = \mathrm{Min}(K, u_i) \text{ for } 1 \leq i \leq s\}$. For each element m in $K \setminus K(\tau)$, in s integers $\langle m, u_1\rangle - a_1, \ldots, \langle m, u_s\rangle - a_s$ at least one is positive. Hence the following holds.

Proposition 1. *For a cone $\tau \neq \{0\}$ in Σ, if $K(\tau) \neq \emptyset$, then $V(\tau) \not\subset B$, and if $|K(\tau)| = 1$, then $\mathrm{orb}(\tau) \cap B = \emptyset$.*

Proposition 2. *Let τ be a cone in Σ. If $\dim \hat{\tau} = \dim \tau + 1$ and $|K(\tau)| \geq 2$, then a generic fiber of $\pi(\tau)$ intersects B at $\max\{\langle m, u\rangle \mid m \in K(\tau)\} - \min\{\langle m, u\rangle \mid m \in K(\tau)\}$ points, where u is one of the two primitive elements in N such that $\tau + \mathbf{R}_{\geq 0}u$ is a cone in Σ contained in $\hat{\tau}$.*

Proof. We can take the above u and a basis $\{u_1, u_2, \ldots, u_r\}$ of N so that $\tau = \mathbf{R}_{\geq 0}u_1 + \cdots + \mathbf{R}_{\geq 0}u_s$, $u_{s+1} = u$. Hence the proposition holds, because the restriction of π to $\mathrm{orb}(\tau)$ is expressed as $(z_{s+1}, \ldots, z_r) \mapsto (z_{s+2}, \ldots, z_r)$, where $z_i = \mathbf{e}^{v_i}$ for the basis $\{v_1, \ldots, v_r\}$ of M dual to $\{u_1, \ldots, u_r\}$. □

Let W and $\widetilde{W}$ be the normalizations of the hypersurfaces

$$\{(y, w) \in Y \times \mathbf{C} \mid P(y) - w^2 = 0\} \quad \text{and} \quad \{(y, w) \in \widetilde{Y} \times \mathbf{C} \mid P(y) - w^2 = 0\}$$

of $Y \times \mathbf{C}$ and $\widetilde{Y} \times \mathbf{C}$, respectively. Let $p : W \to Y$, $\tilde{p} : \widetilde{W} \to \widetilde{Y}$ and $\lambda : \widetilde{W} \to W$ be the natural projections. Then $\tilde{p}$ ramifies along B and $V(\tau)$ for 1-dimensional cones τ in Σ with odd $\mathrm{Min}(K, u_\tau)$. $F := \tilde{p}^{-1}(E)$ is the exceptional set of λ. Let $F_c = \tilde{p}^{-1}(E_c)$.

Theorem 3. *If K and $\{c_m \mid m \in K\}$ satisfy the following conditions (i), (ii) and (iii), then $\widetilde{W}$ is non-singular and F is normal crossing near F_c.*

(i) $K(\tau) \neq \emptyset$ for all cones τ in Σ with $\dim \tau \geq 1$.

(ii) For any 1-dimensional cone τ in Σ, if $\mathrm{Min}(K, u_\tau)$ is odd, then $\tau \not\prec \sigma$, $|K(\tau)| = 1$ and $\mathrm{Min}(K, u_\mu)$ are even for all 1-dimensional cones μ in Σ with $\tau + \mu \in \Sigma(2)$.

(iii) For $\tau \in \Sigma_c$, if $|K(\tau)| > 1$, then $\mathrm{orb}(\tau) \cap B$ ($= \{y \in \mathrm{orb}(\tau) \mid \sum_{m \in K(\tau)} c_m \mathbf{e}^{m - m_0}(y) = 0\}$) is non-singular, where m_0 is an element in $K(\tau)$.

Proof. Irreducible components of the ramification divisor of $\tilde{p}$ do not intersect each other, by the conditions (1), (ii) and Proposition 1. Moreover, B is non-singular near E_c and intersects E transeversally, by the conditions (i) and (iii). □

2 Defining equations

Let σ, Σ, K be as in the previous section. Let

$$J = \{m \in M \mid 2m + K \subset \sigma^\vee, 2m + K \not\subset 2m' + \sigma^\vee \text{ for } m' \in \sigma^\vee \cap M \setminus \{0\}\}.$$

For an element m in M, $2m+K\subset\sigma^\vee$, if and only if $\langle m,u_\tau\rangle\geq-\frac{1}{2}\langle v,u_\tau\rangle$ for all v in K and all 1-dimensional faces τ of σ. Hence $J=\{m\in C\cap M\mid m-m'\notin C \text{ for } m'\in\sigma^\vee\cap M\setminus\{0\}\}$, where

$$C=\{m\in M_{\mathbf{R}}\mid\langle m,u_\tau\rangle\geq-\frac{1}{2}\mathrm{Min}(K,u_\tau)\text{ for }\tau\in\sigma(1)\}$$

and $\sigma(1)$ is the set of 1-dimensional faces of σ.

Proposition 4. *J is a non-empty finite set.*

Proof. Since $C\supset\sigma^\vee$, $J\neq\emptyset$. Let $a_1(\tau)=\min\{\langle m,u_\tau\rangle\mid m\in C\cap M\}$ for each 1-dimensional face τ of σ. Choose an element m_τ in $C\cap M$ with $\langle m_\tau,u_\tau\rangle=a_1(\tau)$ and let $J_1=\{m_\tau\mid\tau\in\sigma(1)\}$. Let $b_1(\tau)=\max\{\langle m,u_\tau\rangle\mid m\in J_1\}$. If an element m_1 in $C\cap M$ satisfies $\langle m_1,u_\tau\rangle\geq b_1(\tau)$ for all 1-dimensional faces τ of σ except one face μ, then m_1 is in $m_\mu+\sigma^\vee$. Hence for each element m_2 in $\left(C\setminus\bigcup_{m\in J_1}m+\sigma^\vee\right)\cap M$, there exist two 1-dimensional faces τ and μ of σ satisfying $\langle m_2,u_\tau\rangle<b_1(\tau)$ and $\langle m_2,u_\mu\rangle<b_1(\mu)$. Let

$$M(\tau,\mu,c,d)=\{m\in C\cap M\mid\langle m,u_\tau\rangle=c,\langle m,u_\mu\rangle=d\}$$

for two 1-dimensional faces τ, μ of σ different to each other and two integers c, d, and let

$$I(\tau,\mu)=\{(c,d)\in\mathbf{Z}^2\mid a_1(\tau)\leq c<b_1(\tau),\ a_1(\mu)\leq d<b_1(\mu),\ M(\tau,\mu,c,d)\neq\emptyset\}.$$

For each element (c,d) in $I(\tau,\mu)$, choose an element $m(\tau,\mu,c,d)$ in $M(\tau,\mu,c,d)$, and let

$$J_2=J_1\bigcup\bigcup_{\tau,\mu\in\sigma(1),\tau\neq\mu}\{m(\tau,\mu,c,d)\mid(c,d)\in I(\tau,\mu)\}.$$

Let $b_2(\tau)=\max\{\langle m,u_\tau\rangle\mid m\in J_2\}$. If an element m_2 in $C\cap M$ satisfies $\langle m_2,u_\tau\rangle\geq b_2(\tau)$ for all 1-dimensional faces τ of σ except two, then $m_2\in\bigcup_{m\in J_2}m+\sigma^\vee$. Repeating these process, we have

$$J\setminus J_{|\sigma(1)|-1}\subset\left(C\setminus\cup_{m\in J_{|\sigma(1)|-1}}m+\sigma^\vee\right)\cap M\subset\{m\in M\mid a_1(\tau)\leq\langle m,u_\tau\rangle<b_{|\sigma(1)|-1}(\tau)\text{ for }\tau\in\sigma(1)\}.$$

Since σ is r-dimensional, $\{m\in M_{\mathbf{R}}\mid a_1(\tau)\leq\langle m,u_\tau\rangle\leq b_{|\sigma(1)|-1}(\tau)\text{ for }\tau\in\sigma(1)\}$ is compact. Hence J is finite. □

In the case $r=2$, we can determine J, as follows: Let $a_0(\tau)=\min\{\langle m,u_\tau\rangle\mid m\in C\cap M\}$ for a 1-dimensional face τ of σ. Let τ, μ be the 1-dimensional faces of σ. Let m_0 be the element in M satisfying $\langle m_0,u_\tau\rangle=a_0(\tau)$ and $\langle m_0,u_\mu\rangle=b_0:=\min\{\langle m,u_\mu\rangle\mid m\in C\cap M,\langle m,u_\tau\rangle=a_0\}$. Then $m_0\in J$ and if $b_0=a_0(\mu)$, then $J=\{m_0\}$. When $b_0\neq a_0(\mu)$, let

$$a_1=\min\{\langle m,u_\tau\rangle\mid m\in C\cap M,\ \langle m,u_\mu\rangle<b_0\},\quad b_1=\min\{\langle m,u_\mu\rangle\mid m\in C\cap M,\ \langle m,u_\tau\rangle=a_1\}.$$

Let m_1 be the element in M satisfying $\langle m_1,u_\tau\rangle=a_1$ and $\langle m_1,u_\mu\rangle=b_1$. Then $m_1\in J$. If $b_1=a_0(\mu)$, then $J=\{m_0,m_1\}$. If $b_1\neq a_0(\mu)$, then repeat this process.

Assume that K and $\{c_m\}_{m\in K}$ satisfy the conditions (i), (ii) and (iii) in Theorem 3 and let $P=\sum_{m\in K}c_m\mathbf{e}^m$. Let $J=\{m_1,\ldots,m_k\}$ and let $P_i=\mathbf{e}^{2m_i}P$. Then $P_i\in\mathbf{C}[\sigma^\vee\cap M]$. Let $Q_{i,j}=\mathbf{e}^{m_i+m_j}P$ for $1\leq i<j\leq k$. Then $Q_{i,j}\in\mathbf{C}[\sigma^\vee\cap M]$, because $m_i+m_j+K\subset\sigma^\vee$. Let

$$L_{i,j}=\{l\in\sigma^\vee\cap M\mid l+m_i-m_j\in\sigma^\vee,\{l,l+m_i-m_j\}\not\subset m+\sigma^\vee\text{ for }m\in\sigma^\vee\cap M\setminus\{0\}\}$$

for $1\leq i<j\leq k$. We can prove the following in a similar manner as in the above proof.

Proposition 5. *$L_{i,j}$ is a non-empty finite set.*

Let

$$W' = \{(y, w_1, \dots, w_k) \in Y \times \mathbf{C}^k \quad | \quad P_i(y) - w_i^2 = 0 \text{ for } 1 \le i \le k,$$
$$Q_{i,j}(y) - w_i w_j = 0 \text{ for } 1 \le i < j \le k,$$
$$\mathbf{e}^l(y)w_i - \mathbf{e}^{l+m_i-m_j}(y)w_j = 0 \text{ for } l \in L_{i,j}, 1 \le i < j \le k\}.$$

In the case $r = 2$, we can eliminate overlappings in the above equations as follows: Let τ, μ be the 1-dimensional faces of σ. We can take sufficies of elements m_i in J so that if $i < j$, then $\langle m_i, u_\tau\rangle < \langle m_j, u_\tau\rangle$ and $\langle m_i, u_\mu\rangle > \langle m_j, u_\mu\rangle$. When $j > i+1$, $\langle l_h, u_\mu\rangle \ge 0$ for $l \in L_{i,j}$, $0 \le h \le j-i$, where $l_h = l+m_i-m_{i+h}$. Moreover, $\langle l_h, u_\tau\rangle \ge 0$, because $\langle l_{j-i}, u_\tau\rangle \ge 0$. Hence there exist elements l'_h and l''_h in $L_{i+h,i+h+1}$ and $\sigma^\vee \cap M$, respectively with $l_h = l'_h + l''_h$ for $0 \le h < j-i$, because l_h is in $\sigma^\vee \cap M$. Since

$$\mathbf{e}^l w_i - \mathbf{e}^{l_{j-i}} w_j = (\mathbf{e}^l w_i - \mathbf{e}^{l_1} w_{i+1}) + (\mathbf{e}^{l_1} w_{i+1} - \mathbf{e}^{l_2} w_{i+2}) + \cdots + (\mathbf{e}^{l_{j-i-1}} w_{j-1} - \mathbf{e}^{l_{j-i}} w_j)$$

and $\mathbf{e}^{l_h} w_{i+h} - \mathbf{e}^{l_{h+1}} w_{i+h+1} = \mathbf{e}^{l''_h}(\mathbf{e}^{l'_h} w_{i+h} - \mathbf{e}^{l'_h + m_{i+h} - m_{i+h+1}} w_{i+h+1})$, we can eliminate $\mathbf{e}^l w_i - \mathbf{e}^{l+m_i-m_j} w_j$ for all elements l in $L_{i,j}$, if $j > i+1$.

Theorem 6. *Above W' is normal and the natural projection $W' \to Y$ is a double covering with the covering transformation $i : (y, w_1, \dots, w_k) \mapsto (y, -w_1, \dots, -w_k)$.*

Proof. Let y be a point in T_N and let w_1 be a complex number with $w_1^2 = P_1(y)$. Then

$$(y, w_1, \frac{\mathbf{e}^{m_2}(y)}{\mathbf{e}^{m_1}(y)} w_1, \dots, \frac{\mathbf{e}^{m_k}(y)}{\mathbf{e}^{m_1}(y)} w_1)$$

is a point on W' and any point on $W'_o := (T_N \times \mathbf{C}^k) \cap W'$ is expressed in this way. Hence it suffices to show that W' is normal.

Let $q : \overline{W'} \to W'$ be the normalization of W' and let $\bar{i}$ be the automorphism of $\overline{W'}$ with $q \circ \bar{i} = i \circ q$. In the following, we show that for any holomorphic function f on $\overline{W'}$ there exist elements f_0, f_1, …, f_k in $\mathbf{C}\{\sigma^\vee \cap M\}$ with $f = f_0 + \sum_{i=1}^k f_i w_i$. Let $f_0 = \frac{1}{2}(f + \bar{i}^* f)$ and let $f' = \frac{1}{2}(f - \bar{i}^* f)$. Then f_0 is in $\mathbf{C}\{\sigma^\vee \cap M\}$ and $\bar{i}^* f' = -f'$. Hence $f'/q^* w_1$ is a meromorphic function on Y with poles only along $Y \setminus T_N$. Therefore, there exists an element m_0 in $\sigma^\vee \cap M$ such that $\mathbf{e}^{m_0} f'/q^* w_1$ is holomorphic. Hence there exist a subset M' of M and non-zero complex numbers c_m for m in M' such that $f'/q^* w_1 = \sum_{m \in M'} c_m \mathbf{e}^m$. Let m be an element in M'. Since $\mathbf{e}^{2m} P_1$ is in $\mathbf{C}[\sigma^\vee \cap M]$, $2m + 2m_1 + K \subset \sigma^\vee$. Hence $m + m_1 = l + m_i$ for elements m_i in J and l in $\sigma^\vee \cap M$. Therefore, there exist subsets M'_i of M' for $1 \le i \le k$ such that $M' = M'_1 \sqcup \cdots \sqcup M'_k$ and $m + m_1 - m_i$ is in $\sigma^\vee$ for any element m in M'_i. Let $f_i = \sum_{m \in M'_i} c_m \mathbf{e}^{m+m_1-m_i}$. Then f_i is in $\mathbf{C}\{\sigma^\vee \cap M\}$ and $f' = \sum_{i=1}^k f_i w_i$.

Let I be the ideal of $\mathbf{C}[\sigma^\vee \cap M][w_1, \dots, w_k]$ generated by $P_i - w_i^2$ for $1 \le i \le k$, $Q_{i,j} - w_i w_j$ for $1 \le i < j \le k$ and $\mathbf{e}^l w_i - \mathbf{e}^{l+m_i-m_j} w_j$ for $l \in L_{i,j}$, and let f be an element in $\mathbf{C}[\sigma^\vee \cap M][w_1, \dots, w_k]$. We show that if $f \equiv 0$ on $W'_o = (T_N \times \mathbf{C}^k) \cap W'$, then f is in I. Since $P_i - w_i^2, Q_{i,j} - w_i w_j \in I$, there exist elements g in I and f_0, …, f_k in $\mathbf{C}[\sigma^\vee \cap M]$ with $f - g = f_0 + f_1 w_1 + \cdots + f_k w_k$. Clearly, $f_0 = 0$ and $f' := f_1 w_1 + \cdots + f_k w_k \equiv 0$ on W'_0. Since $\mathbf{e}^{m_1} w_i = \mathbf{e}^{m_i} w_1$ on W'_0, $\mathbf{e}^{m_1} w_1 f' = P_1 \sum_{i=1}^k \mathbf{e}^{m_i} f_i$. Hence $\sum_{i=1}^k c_{i,v-m_i} = 0$ for each v in $\sigma^\vee \cap M$, where $c_{i,v}$ are coefficients in the sum $f_i = \sum c_{i,v} \mathbf{e}^v$. On the other hand, if $v_i + m_i = v_j + m_j$ for elements v_i, v_j in $\sigma^\vee \cap M$, then $\mathbf{e}^{v_i} w_i - \mathbf{e}^{v_j} w_j$ is in I. Hence f' is in I. □

The ramification divisor of the double covering $W' \to Y$ is equal to that of $p : W \to Y$. Hence W' is biholomorphic to W. Clearly, the following holds.

Proposition 7. *$J = \{0\}$ if and only if $\operatorname{Min}(K, u_\tau) = 0$ for all 1-dimensional faces τ of σ.*

If the condition in the above proposition holds, then $\{(y, w) \in Y \times \mathbf{C} \mid P - w^2 = 0)\}$ is normal, by Theorem 6.

3 Simple elliptic singularities and two dimensional cusp singularities

In this section, we restrict ourselves to the case $r = 2$. A simple elliptic singularity is a two dimensional singularity the exceptional set of whose minimal resolution consists of an elliptic curve and the complex structure depends only on that of the curve and the self-intersection number (see [1]).

Theorem 8. *Any simple elliptic singularity with $E^2 < -2$ is a double covering of a toric singularity, where E is the exceptional set of the minimal resolution. If E^2 is even and E is biholomorphic to the double covering of $\mathbf{P}^1$ ramifying at 0, 1, ∞, τ, then the defining equations are*

$$z_i z_j - z_{i+1} z_{j-1} = 0 \ (1 \le i, i+2 \le j \le a+1), \quad z_i^2 + \gamma z_{i+1}^2 + z_{i+2}^2 - w_i^2 = 0 \ (1 \le i \le a-1),$$

$$z_i z_j + \gamma z_{i+1} z_{j+1} + z_{i+2} z_{j+2} - w_i w_j = 0 \ (1 \le i < j \le a-1), \ z_{k+1} w_i - z_k w_{i+1} = 0 \ (1 \le i \le a-2, 1 \le k \le a),$$

where $a = -\frac{1}{2}E^2$ and $\gamma = -2(1+\tau)/(1-\tau)$. If E^2 is odd and E is biholomorphic to the double covering of $\mathbf{P}^1$ ramifying at 0, 1, ∞, τ, then the defining equations are

$$z_1 z_i - z_2^2 z_{i-1} = 0 \ (3 \le i \le a+1), \ z_i z_j - z_{i+1} z_{j-1} = 0 \ (2 \le i, i+2 \le j \le a+1), \ z_1 + \gamma z_2^2 + z_3^2 - w_1^2 = 0,$$

$$z_i z_{i+1} + \gamma z_{i+1}^2 + z_{i+2}^2 - w_i^2 = 0 \ (2 \le i \le a-1), \ z_{i+1} z_j + \gamma z_{i+1} z_{j+1} + z_{i+2} z_{j+2} - w_i w_j = 0 (1 \le i < j \le a-1),$$

$$z_2^2 w_i - z_1 w_{i+1} = 0 \ (1 \le i \le a-2), \ z_{k+1} w_i - z_k w_{i+1} = 0 \ (1 \le i \le a-2, 2 \le k \le a),$$

where $a = \frac{1}{2}(-E^2+1)$ and α, β are complex numbers satisfying $\beta(\alpha+\beta)+(\alpha+\beta)(\alpha\tau+\beta)+(\alpha\tau+\beta)\beta = 0$, $\beta(\alpha+\beta)(\alpha\tau+\beta) = -1$ and $\gamma = -(1+\tau)\alpha - 3\beta$.

Proof. Let a be an integer greater than 1. Let $\sigma = \mathbf{R}_{\ge 0}{}^t(-1,1) + \mathbf{R}_{\ge 0}{}^t(1, a-1)$ and let

$$\Sigma = \{\text{faces of } \mathbf{R}_{\ge 0}{}^t(-1,1) + \mathbf{R}_{\ge 0}{}^t(0,1) \text{ and } \mathbf{R}_{\ge 0}{}^t(0,1) + \mathbf{R}_{\ge 0}{}^t(1,a-1)\}.$$

Then $K = \{(2,2),(0,2),(-2,2)\}$ satisfies the conditions (i) and (ii) in Theorem 3. $\mathrm{Min}(K, u_\tau)$ are even for all 1-dimensional cones τ in Σ. $|K(\tau)| = 1$ for all cones τ in Σ except $\mathbf{R}_{\ge 0}{}^t(0,1)$. We see by Proposition 2 that B intersects $V(\mathbf{R}_{\ge 0}{}^t(0,1))$ at 4 points. Hence $F_{\mathrm{c}} = \tilde{p}^{-1}(V(\mathbf{R}_{\ge 0}{}^t(0,1)))$ is an elliptic curve with the self-intersection numer $-2a$. $\sigma^\vee \cap M$ is generated by $(1,1)$, $(0,1)$, …, $(-a+1,1)$. Hence the defining equations of Y are $z_i z_j - z_{i+1} z_{j-1} = 0$ for $1 \le i, i+2 \le j \le a+1$, where $z_i = \mathbf{e}^{(-i+2,1)}$ for $1 \le i \le a+1$. We easily see that $J = \{(0,0),(-1,0),\ldots,(-a+2,0)\}$ and $L_{i,i+1} = \{(0,1),\ldots,(-a+1,1)\}$. Let τ_0, μ be complex numbers satisfying $\tau_0^2 = \tau$, $\mu^2 = (1+\tau_0)/(1-\tau_0)$ and let γ be as in the theorem. Then the transformation $z \mapsto \mu(z-\tau_0)/(z+\tau_0)$ maps $\{0,1,\infty,\tau\}$ to the roots of $z^4 + \gamma z^2 + 1 = 0$. Let $c_{(2,2)} = c_{(-2,2)} = 1$, $c_{(0,2)} = \gamma$. Since $z^4 + \gamma z^2 + 1 = 0$ has no multiple root, the condition (iii) is satisfied and the exceptional set F_c of λ is biholomorphic to the double covering of $\mathbf{P}^1$ ramifying at 0, 1, ∞, τ.

Let $\sigma = \mathbf{R}_{\ge 0}{}^t(-2,1) + \mathbf{R}_{\ge 0}{}^t(1,a-1)$ and let

$$\Sigma = \{\text{faces of } \mathbf{R}_{\ge 0}{}^t(-2,1) + \mathbf{R}_{\ge 0}{}^t(-1,1), \mathbf{R}_{\ge 0}{}^t(-1,1) + \mathbf{R}_{\ge 0}{}^t(0,1) \text{ and } \mathbf{R}_{\ge 0}{}^t(0,1) + \mathbf{R}_{\ge 0}{}^t(1,a-1)\}.$$

Then $K = \{(1,2),(0,2),(-2,2)\}$ satisfies the conditions (i), (ii) and $\mathrm{Min}(K, u_\tau)$ are even for all 1-dimensional cones τ in Σ except $\mathbf{R}_{\ge 0}{}^t(-1,1)$. We see by Proposition 2 that the exceptional set $F_{\mathrm{c}} = \tilde{p}^{-1}(V(\mathbf{R}_{\ge 0}{}^t(-1,1)) + V(\mathbf{R}_{\ge 0}{}^t(0,1)))$ of λ consists of an elliptic curve with the self-intersection number $-2a$ and a rational curve with the self-intersection number -1 intersecting at a point. Hence the exceptional set of the minimal resolution is an elliptic curve with the self-intersection number $-2a+1$. $\sigma^\vee \cap M$ is generated by $(1,2)$, $(0,1)$, $(-1,1)$, …, $(-a+1,1)$. Hence the defining equations of Y are $z_1 z_i - z_2^2 z_{i-1} = 0$ for $3 \le i \le a+1$ and $z_i z_j - z_{i+1} z_{j-1} = 0$ for $2 \le i, i+2 \le j \le a+1$, where $z_1 = \mathbf{e}^{(1,2)}$, $z_i = \mathbf{e}^{(-i+2,1)}$ for $2 \le i \le a+1$. We easily see that $J = \{(0,0),(-1,0),\ldots,(-a+2,0)\}$ and

$L_{i,i+1} = \{(0,2),(-1,1),\ldots,(-a+1,1)\}$. Let α, β, γ be as in the theorem. Then the transformation $z \mapsto \alpha z + \beta$ maps ∞ to itself and $\{0,1,\tau\}$ to the roots of $z^3+\gamma z^2+1=0$. Let $c_{(1,2)} = c_{(-2,2)} = 1$, $c_{(0,2)} = \gamma$. Since $z^3+\gamma z^2+1=0$ has no multiple root, the condidtion (iii) is satisfied and the exceptional set is biholomorphic to the double covering of $\mathbf{P}^1$ ramifying at 0, 1, ∞, τ. □

The exceptional set of the minimal resolution of a 2-dimensional cusp singularity is a cycle $E_1+\cdots+E_k$ of rational curves or a rational curve E_1 with a node. Here if $k>2$, then $E_i\cdot E_{i+1}=1$ for $1\le i<k$ and $E_k\cdot E_1=1$. While, if $k=2$, then $E_1\cdot E_2=2$. The complex structure depends only on the self-intersection numbers E_i^2. Let $a_1=E_1^2-2$, if $k=1$ and let $a_i=E_i^2$, if $k>1$.

Definition. We say that a 2-dimensional cusp singularity is symmetric, if there exists a map $g:\mathbf{Z}_k\to\mathbf{Z}_k$ satisfying $g^2=\mathrm{id}$, $g(i+1)=g(i)-1$ and $a_{g(i)}=a_i$, where $\mathbf{Z}_k=\mathbf{Z}/k\mathbf{Z}$.

If $k=1$ or 2, then any cusp singularity is symmetric.

Proposition 9. *Any symmetric* 2*-dimensional cusp singularity is a double covering of a toric singularity.*

Proof. In the case $k>2$, we may assume that k is even and $a_i=a_{k+2-i}$ for $2\le i\le k/2$, if necessary shifting the suffices of a_1, …, a_k, and inserting -1 and reducing two integers on both sides, i.e., changing …, a_{i-1}, a_i, a_{i+1}, a_{i+2}, … to …, a_{i-1}, a_i-1, -1, $a_{i+1}-1$, a_{i+2}, … . In the case $k=1$, we change a_1 to -1, a_1-2. Then all integers except a_1 and $a_{k/2+1}$ are smaller than -1 and if $a_1=-1$ (resp. $a_{k/2+1}=-1$), then $a_2<-2$ (resp. $a_{k/2}<-2$).

Let $l=k/2+1$, $k/2+2$ or $k/2+3$, accordingly as the number of odd integers in $\{a_1,a_{k/2+1}\}$ is equal to 0, 1 or 2. We define the sequence b_1, …, b_l as follows: If a_1 is even (resp. odd), then $b_1=a_1/2$, $b_2=a_2$, … (resp. $b_1=-2$, $b_2=(a_1-1)/2$, $b_3=a_2$, …). If $a_{k/2+1}$ is even (resp. odd), then $b_l=a_{k/2+1}/2$, $b_{l-1}=a_{k/2}$, … (resp. $b_l=-2$, $b_{l-1}=(a_{k/2+1}-1)/2$, $b_{l-2}=a_{k/2}$, …). There exist elements u_0, u_1, …, u_{l+1} of $N=\mathbf{Z}^2$ such that $\{u_i,u_{i+1}\}$ are bases of N for $0\le i\le l$ and that $u_{i-1}+b_iu_i+u_{i+1}=0$ for $1\le i\le l$. Then $\sigma=\mathbf{R}_{\ge0}u_0+\mathbf{R}_{\ge0}u_{l+1}$ is a strongly convex cone and

$$\Sigma=\{\{0\},\mathbf{R}_{\ge0}u_i,\mathbf{R}_{\ge0}u_j+\mathbf{R}_{\ge0}u_{j+1}\mid 0\le i\le l+1, 0\le j\le l\}$$

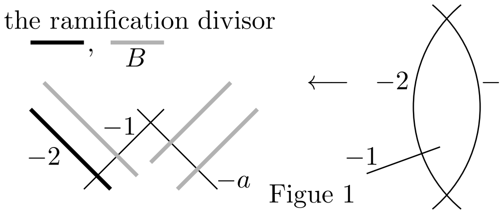
is a non-singular fan with $|\Sigma|=\sigma$. Let m_0 be an element in $2M$ with $\langle m_0,u_0\rangle\ge2$, $\langle m_0,u_{l+1}\rangle\ge2$. Let m_1 be the element in M such that $\langle m_1,u_0\rangle=\langle m_0,u_0\rangle-2$, $\langle m_1,u_1\rangle=\langle m_0,u_1\rangle$ (resp. $\langle m_1,u_2\rangle=\langle m_0,u_2\rangle$), if a_1 is even (resp. odd), and let m_2 be the element in M such that $\langle m_2,u_{l+1}\rangle=\langle m_0,u_{l+1}\rangle-2$, $\langle m_2,u_l\rangle=\langle m_0,u_l\rangle$ (resp. $\langle m_2,u_{l-1}\rangle=\langle m_0,u_{l-1}\rangle$), if $a_{k/2+1}$ is even (resp. odd). Then $K=\{m_0,m_1,m_2\}$ satisfies the conditions (i), (ii) in Theorem 3. If a_1 is even (resp. odd), then $\mathrm{Min}(K,u_1)$ is even (resp. odd). If $a_{k/2+1}$ is even (resp. odd), then $\mathrm{Min}(K,u_l)$ is even (resp. odd). $\mathrm{Min}(K,u_i)$ are even for $i\ne1,l$. $|K(\tau)|>1$ only for $\tau=\mathbf{R}_{\ge0}u_1$ (resp. $\mathbf{R}_{\ge0}u_2$) if a_1 is even (resp. odd) and for $\tau=\mathbf{R}_{\ge0}u_l$ (resp. $\mathbf{R}_{\ge0}u_{l-1}$) if $a_{k/2+1}$ is even (resp. odd). Let $E_i=V(\mathbf{R}_{\ge0}u_i)$. $c_{m_0}=c_{m_1}=c_{m_2}=1$ satisfies the condition (iii). If a_1 is even, then B intersects E_1 at 2 points. If a_1 is odd, then B intersects E_2 at 1 point. If $a_{k/2+1}$ is even, then B intersects E_l at 2 points. If $a_{k/2+1}$ is odd, then B intersects E_{l-1} at 1 point. Moreover, $\tilde{p}^{-1}(E_1)$ (resp. $\tilde{p}^{-1}(E_l)$) is a exceptional curve of the first kind and hence contractible, if a_1 (resp. $a_{k/2+1}$) is odd. Thus we obtain a desired double covering. □

We give examples for $k=1$ and $a_1<-4$.

Example 1. Let a_1 be an even integer smaller than -5 and let $a=-a_1/2+1$. Then $l=3$, $b_1=-2$, $b_2=-1$, $b_3=-a$ in the above proof and we can take

$$u_0=\begin{pmatrix}-1\\1\end{pmatrix},\ u_1=\begin{pmatrix}-1\\2\end{pmatrix},\ u_2=\begin{pmatrix}-1\\3\end{pmatrix},\ u_3=\begin{pmatrix}0\\1\end{pmatrix},\ u_4=\begin{pmatrix}1\\a-3\end{pmatrix},$$

$K=\{m_0=(0,2),m_1=(3,3),m_2=(-2,2)\}$. $\mathrm{Min}(K,u_\tau)$ are even for all 1-dimensional cones τ in Σ except $\mathbf{R}_{\ge0}u_1$. B intersects $V(\mathbf{R}_{\ge0}u_2)$ at 1 point and $V(\mathbf{R}_{\ge0}u_3)$ at 2 points(see Figure 1). If we contracts the exceptional curves of the first kind in F_c, then F_c becomes a rational curve with a node and the self-intersection number is $-2a+4$. $\sigma^\vee \cap M$ is generated by $v_1 = (1,1)$, $v_2 = (0,1)$, …, $v_{a-1} = (-a+3,1)$. Hence the defining equations of Y are $z_i z_j - z_{i+1} z_{j-1} = 0$ for $1 \le i,\ i+2 \le j \le a-1$, where $z_i = \mathbf{e}^{v_i}$. We easily see that $J = \{(0,0),(-1,0),\ldots,(-a+4,0)\}$ and $L_{i,i+1} = \{v_2,\ldots,v_{a-1}\}$. Hence the defining equations of W' are the above ones for Y and $z_1 z_i^2 + z_{i+1}^2 + z_{i+2}^2 - w_i^2 = 0$ for $1 \le i \le a-3$, $z_1 z_i z_j + z_{i+1} z_{j+1} + z_{i+2} z_{j+2} - w_i w_j = 0$ for $1 \le i < j \le a-3$, $z_{k+1} w_i - z_k w_{i+1} = 0$ for $1 \le i \le a-4$, $1 \le k \le a-2$.



Figue 1

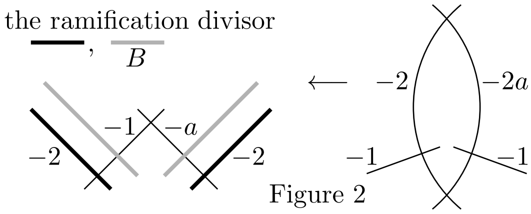


Figure 2

Example 2. Let a_1 be an odd integer smaller than -4 and let $a = (-a_1+3)/2$. Then $l = 4$, $b_1 = -2$, $b_2 = -1$, $b_3 = -a$, $b_4 = -2$ in the above proof and we can take

$$u_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ u_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix},\ u_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix},\ u_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ u_4 = \begin{pmatrix} 1 \\ a-3 \end{pmatrix},\ u_5 = \begin{pmatrix} 2 \\ 2a-7 \end{pmatrix},$$

$K = \{m_0 = (0,2), m_1 = (3,3), m_2 = (-1,2)\}$. $\mathrm{Min}(K, u_\tau)$ are even for all 1-dimensional cones τ in Σ except $\mathbf{R}_{\ge 0} u_1$ and $\mathbf{R}_{\ge 0} u_4$. B intersects $V(\mathbf{R}_{\ge 0} u_2)$ and $V(\mathbf{R}_{\ge 0} u_3)$ at 1 point(see Figure 2). If we contract the exceptional curves of the first kind in F_c, then F_c becomes a rational curve with a node and the self-intersection number is $-2a+5$. $\sigma^\vee \cap M$ is generated by $v_1 = (1,1)$, $v_2 = (0,1)$, $\cdots$, $v_{a-2} = (-a+4,1)$, $v_{a-1} = (-2a+7,2)$. Hence the defining equations of Y are $z_i z_j - z_{i+1} z_{j-1} = 0$ for $1 \le i, i+2 \le j \le a-2$, $z_i z_{a-1} - z_{i+1} z_{a-2}^2 = 0$ for $1 \le i \le a-3$, where $z_i = \mathbf{e}^{v_i}$. We easily see that $J = \{(0,0),(-1,0),\ldots,(-a+4,0)\}$, and $L_{i,i+1} = \{v_2,\ldots,v_{a-2},v_{a-1}\}$. Hence the defining equations of W' are the above ones for Y and $z_1 z_i^2 + z_{i+1}^2 + z_{i+1} z_{i+2} - w_i^2 = 0$ for $1 \le i \le a-4$, $z_1 z_{a-3}^2 + z_{a-2}^2 + z_{a-1} - w_{a-3}^2 = 0$, $z_1 z_i z_j + z_{i+1} z_{j+1} + z_{i+2} z_{j+1} - w_i w_j = 0$ for $1 \le i < j \le a-3$, $z_{k+1} w_i - z_k w_{i+1} = 0$ for $1 \le i \le a-4$, $1 \le k \le a-3$, $z_{a-1} w_i - z_{a-2}^2 w_{i+1} = 0$ for $1 \le i \le a-4$.

Some of the quotients by finite cyclic groups are also double coverings of toric singularities.

Example 3. Let a be an integer greater than 1. Let $\sigma = \mathbf{R}_{\ge 0} u_1 + \mathbf{R}_{\ge 0} u_5$ and let

$$\Sigma = \{\{0\}, \mathbf{R}_{\ge 0} u_i, \mathbf{R}_{\ge 0} u_j + \mathbf{R}_{\ge 0} u_{j+1} \mid 1 \le i \le 5, 1 \le j \le 4\},$$

where

$$u_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix},\ u_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ u_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ u_4 = \begin{pmatrix} 1 \\ a-1 \end{pmatrix}\ u_5 = \begin{pmatrix} 2 \\ 2a-3 \end{pmatrix}.$$

Then Σ is a non-singular fan with $|\Sigma| = \sigma$ and $K = \{(0,4),(2,4)\}$ satisfies the conditions (i), (ii) in Theorem 3. The exceptional set F is as in Figure 3. If $a = 2$, then $\sigma^\vee \cap M$ is generated by $v_1 = (1,2)$, $v_2 = (0,1)$, $v_3 = (-1,2)$, and $J = \{(0,0),(-1,0)\}$, $L_{1,2} = \{(0,2),(-1,2)\}$. If we set $c_{(0,4)} = c_{(2,4)} = 1$, then $P_1 = z_1^2 + z_2^4$, $P_2 = z_2^4 + z_3^2$, $Q_{1,2} = z_1 z_2^2 + z_2^2 z_3$, and the defining equations of W' are $z_1 z_3 - z_2^4 = 0$, $P_i - w_i^2 = 0$ for $i = 1,2$, $Q_{1,2} - w_1 w_2 = 0$, $z_2^2 w_1 - z_1 w_2 = 0$, $z_3 w_1 - z_2^2 w_2 = 0$, where $z_i = \mathbf{e}^{v_i}$.

Example 4. Let a and b be integers greater than 1. Let $\sigma = \mathbf{R}_{\ge 0} u_1 + \mathbf{R}_{\ge 0} u_7$ and let $\Sigma = \{\{0\}, \mathbf{R}_{\ge 0} u_i, \mathbf{R}_{\ge 0} u_j + \mathbf{R}_{\ge 0} u_{j+1} \mid 1 \le i \le 7, 1 \le j \le 6\}$, where

$$u_1 = \begin{pmatrix} -2b+1 \\ 2b-3 \end{pmatrix},\ u_2 = \begin{pmatrix} -b \\ b-1 \end{pmatrix},\ u_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ u_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$u_5 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ u_6 = \begin{pmatrix} a \\ a-1 \end{pmatrix},\ u_7 = \begin{pmatrix} 2a-1 \\ 2a-3 \end{pmatrix}.$$

Then Σ is a non-singular fan with $|\Sigma| = \sigma$ and $K = \{(-2,6),(-1,5),(0,6)\}$ satisfies the conditions (i), (ii) in Theorem 3. The exceptional set is as in Figure 4. If $a = b = 2$, then $\sigma^\vee \cap M$ is generated by $v_1 = (-1,3)$, $v_2 = (0,1)$, $v_3 = (1,3)$ and $J = \{(0,0),(1,0)\}$, $L_{1,2} = \{(0,3),(1,3)\}$. If we set $c_{(-2,6)} = c_{(-1,5)} = c_{(0,6)} = 1$, then $P_1 = z_1^2 + z_1 z_2^2 + z_2^6$, $P_2 = z_2^6 + z_2^2 z_3 + z_3^2$, $Q_{1,2} = z_1 z_2^3 + z_2^5 + z_2^3 z_3$, where $z_i = \mathbf{e}^{v_i}$. The defining equations of W' are $z_1 z_3 - z_2^6 = 0$, $P_i - w_i^2 = 0$ for $i = 1,2$, $Q_{1,2} - w_1 w_2 = 0$, $z_2^3 w_1 - z_1 w_2 = 0$, $z_3 w_1 - z_2^3 w_2 = 0$.

Example 5. Let a be a positive integer. Let $\sigma = \mathbf{R}_{\geq 0} u_1 + \mathbf{R}_{\geq 0} u_6$ and let $\Sigma = \{\{0\}, \mathbf{R}_{\geq 0} u_i, \mathbf{R}_{\geq 0} u_j + \mathbf{R}_{\geq 0} u_{j+1} \mid 1 \leq i \leq 6, 1 \leq j \leq 5\}$, where

$$u_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix},\ u_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix},\ u_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ u_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ u_5 = \begin{pmatrix} 1 \\ a-1 \end{pmatrix}\ u_6 = \begin{pmatrix} 6 \\ 6a-7 \end{pmatrix}.$$

Then Σ is a non-singular fan with $|\Sigma| = \sigma$ and $K = \{(2,6),(1,6)\}$ satisfies the conditions (i), (ii) in Theorem 3. The exceptional set F is as in Figure 5. If $a = 1$, then $\sigma^\vee \cap M$ is generated by $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, and $J = \{(0,0)\}$. If we set $c_{(2,6)} = c_{(1,6)} = 1$, $z_i = \mathbf{e}^{(1,2+i)}$, then the defining equations are $z_1 z_3 - z_2^2 = 0$, $z_1 z_4 - z_2 z_3 = 0$, $z_2 z_4 - z_3^2 = 0$, $z_1^2 + z_4 - w^2 = 0$.

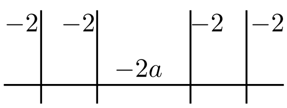


Figure 3

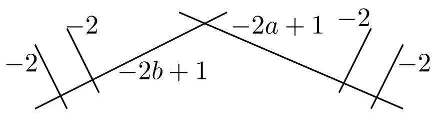


Figure 4

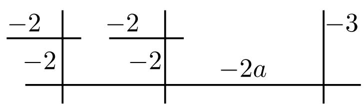


Figure 5

4 Log canonical singularities

Let σ, Σ, K, $\{c_m\}_{m \in K}$, $p : W \to Y$, $\tilde{p} : \widetilde{W} \to \widetilde{Y}$ and $\mathrm{Min}(K,u)$ be as in Section 1. Assume that K and $\{c_m\}_{m \in K}$ satisfy the conditions (1), (ii) and (iii) in Theorem 3.

Theorem 10. *If Σ and K satisfy the following condition (iv), then W is $\mathbf{Q}$-Gorenstein near $p^{-1}(y_0)$.*
(iv) There exists an element m_0 in $M_{\mathbf{Q}}$ with $\langle m_0, u_\tau \rangle = 2 + \mathrm{Min}(K, u_\tau)$ for all 1-dimensional faces τ of σ.
Moreover, W is log canonical, if the following condition (v) holds.
(v) $\langle m_0, u_\tau \rangle \geq \mathrm{Min}(K, u_\tau)$ for all 1-dimensional cones τ in Σ.

Proof. Let $(z_1, z_2, \ldots, z_r)$ be a global coordinate of T_N and let $\omega = (dz_1/z_1) \wedge \cdots \wedge (dz_r/z_r)$. Then ω has poles of order 1 along $\widetilde{Y} \setminus T_N$. $\tilde{p}^* \left(\omega^{\otimes 2}/P\right)$ is a double r form, which is holomorphic and nowhere vanishing on $\tilde{p}^{-1}(T_N)$. Let k be a positive integer with $k m_0 \in M$. If $\mathrm{Min}(K, u_\tau)$ is even (resp. odd) for a 1-dimensional cone τ in Σ, then $\bar{\omega} := \tilde{p}^* \left(\mathbf{e}^{k m_0} \omega^{\otimes 2k}/P^k\right)$ has zeros of order

$$k\langle m_0, u_\tau \rangle - 2k - k\mathrm{Min}(K, u_\tau) \quad (\text{resp. } 2k\langle m_0, u_\tau \rangle - 2k - 2k\mathrm{Min}(K, u_\tau))$$

along $\tilde{p}^{-1}(\mathrm{orb}(\tau))$. Hence $\bar{\omega}$ is holomorphic and nowhere vanishing on $\widetilde{W} \setminus \tilde{p}^{-1}(E)$, if (iv) holds. Moreover, $\bar{\omega}$ has poles of order at most $2k$ along $\tilde{p}^{-1}(\mathrm{orb}(\tau))$, if (v) holds. □

All examples in the previous section satisfy the above conditions (iv) and (v).

Example 6. Let $r = 3$ and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of N. Let a be an integer greater than 1 and let $\sigma = \mathbf{R}_{\geq 0}(a\mathbf{e}_1 + \mathbf{e}_3) + \mathbf{R}_{\geq 0}(a\mathbf{e}_2 + \mathbf{e}_3) + \mathbf{R}_{\geq 0}\mathbf{e}_3$. Then $\sigma^\vee \cap M$ is generated by $\mathbf{e}_1^*$, $\mathbf{e}_2^*$, $\mathbf{e}_3^*$ and

$-\mathbf{e}_1^* - \mathbf{e}_2^* + a\mathbf{e}_3^*$, where $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$ is the basis of M dual to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Hence the defining equation of Y is $z_1 z_2 z_3 - z_0^a = 0$, where $z_0 = \mathbf{e}^{\mathbf{e}_3^*}$, $z_1 = \mathbf{e}^{\mathbf{e}_1^*}$, $z_2 = \mathbf{e}^{\mathbf{e}_2^*}$ and $z_3 = \mathbf{e}^{-\mathbf{e}_1^* - \mathbf{e}_2^* + a\mathbf{e}_3^*}$. Let

$$\begin{aligned}\Sigma &= \{\text{faces of } \mathbf{R}_{\geq 0}\mathbf{f}(i,j) + \mathbf{R}_{\geq 0}\mathbf{f}(i+1,j) + \mathbf{R}_{\geq 0}\mathbf{f}(i,j+1) \mid 0 \leq i, 0 \leq j, i+j \leq a-1\} \\ &\quad \bigcup \{\text{faces of } \mathbf{R}_{\geq 0}\mathbf{f}(i,j) + \mathbf{R}_{\geq 0}\mathbf{f}(i-1,j) + \mathbf{R}_{\geq 0}\mathbf{f}(i,j-1) \mid 1 \leq i, 1 \leq j, i+j \leq a\},\end{aligned}$$

where $\mathbf{f}(i,j) = i\mathbf{e}_1 + j\mathbf{e}_2 + \mathbf{e}_3$. Then Σ is a non-singular fan with $|\Sigma| = \sigma$. Let $K = \{2\mathbf{e}_3^*, 2\mathbf{e}_1^* + 2\mathbf{e}_2^*, 2a\mathbf{e}_3^* - 2\mathbf{e}_1^*, 2a\mathbf{e}_3^* - 2\mathbf{e}_2^*\}$ and let $c_m = 1$ for all elements m in K. Then the conditions (i), (ii) and (iii) in Theorem 3 hold, and $m_0 = 2\mathbf{e}_3^*$ satisfies the conditions (iv) and (v) in the above theorem. Hence W is log canonical. By Proposition 7 and Theorem 6, the defining equations of W are the above one for Y and $z_0^2 + z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2 - w^2 = 0$.

5 Plurigenera of isolated singularities

We keep the notations and the assumptions in the previous section.

Proposition 11. *$p^{-1}(y_0)$ is an isolated singularity of W, if all proper faces of σ are non-singular and elements in Σ.*

Proof. y_0 is an isolated singularity of Y and the restriction of π to $\widetilde{Y} \setminus \pi^{-1}(y_0)$ is biholomorphic, by the assumption. Since the branch locus of the restriction of $\tilde{p}$ to $\tilde{p}^{-1}(U)$ is non-singular for a small enough neighborhood U of $\pi^{-1}(y_0)$, so is $\tilde{p}^{-1}(U \setminus \pi^{-1}(y_0))$. □

We recall the definition of the plurigenra $\delta_l(V,p)$, $d_l(V,p)$ and $\kappa_\delta(V,p)$ defined in [2] and [5] for an isolated singularity (V,p). Let $\pi\colon (U,E) \to (V,p)$ be a resolution and assume that the exceptional set E is normal crossing. Let

$$\delta_l(V,p) = \dim H^0(V \setminus \{p\}, \mathcal{O}_V(lK_V))/H^0(U, \mathcal{O}_U(lK_U + (l-1)E)),$$

$$d_l(V,p) = \dim H^0(U, \mathcal{O}_U(lK_U + lE))/H^0(U, \mathcal{O}_U(lK_U + (l-1)E))$$

for positive integers l, where K_V and K_U are the canonical divisors of V and U, respectively. $\kappa_\delta(V,p) = -\infty$, if $\delta_l(V,p) = 0$ for all l. In the other cases, κ_δ and κ_d are increasing order of $\delta_l(V,p)$ and $d_l(V,p)$, respectively. $\delta_l(V,p) = d_l(V,p)$, if $\kappa_d \leq r-2$.

In the following, we assume that $p^{-1}(y_0)$ is an isolated singularity of W. We denote by $\sigma(i)$ the set of i-dimensional faces of σ. For an element μ in $\Sigma(1) \setminus \sigma(1)$, let

$$S(\mu) = \{v \in M_{\mathbf{R}} \mid \langle v, u_\tau \rangle \geq 1 + \frac{1}{2}\mathrm{Min}(K, u_\tau) \text{ for } {}^\forall\tau \in \sigma(1),\ \langle v, u_\mu \rangle \leq \frac{1}{2}\mathrm{Min}(K, u_\mu)\}$$

and let $S = \bigcup_{\mu \in \Sigma(1)\setminus\sigma(1)} S(\mu)$. Note that $S(\mu)$ is a bounded convex set whose vertices are rational points, if non-empty.

Theorem 12. *If $S = \emptyset$, then $\kappa_\delta(W, p^{-1}(y_0)) = -\infty$. If $S \neq \emptyset$, then $\kappa_\delta(W, p^{-1}(y_0)) = \dim S$. If $1 \leq \dim S < r$, then $\delta_l(W, p^{-1}(y_0)) = d_l(W, p^{-1}(y_0)) = |lS \cap M|$. In particular, $p_g = |S \cap M|$.*

Proof. Let w be a function on $\widetilde{W}$ with $w^2 = \tilde{p}^* P$. Let $(z_1, z_2, \ldots, z_r)$ be a global coordinate of T_N and let $\omega = (dz_1/z_1) \wedge \cdots \wedge (dz_r/z_r)$. Then $\tilde{p}^*\omega^{\otimes l}/w^l$ is an l-ple r-form which is nowhere vanishing on $\tilde{p}^{-1}(T_N)$ and has poles only along $\widetilde{W} \setminus \tilde{p}^{-1}(T_N)$. Hence $\theta w^l/\tilde{p}^*\omega^{\otimes l}$ is a holomorphic function on $W \setminus \{p^{-1}(y_0)\}$ for an element θ in $H^0(W \setminus p^{-1}(y_0), \mathcal{O}_W(lK_W))$. If $\mathrm{Min}(K, u_\tau)$ is even (resp. odd) for $\tau \in \Sigma(1)$, then $\mathbf{e}^v \tilde{p}^*\omega^{\otimes l}/w^l$ has zeros of order $\langle v, u_\tau \rangle - l - \frac{l}{2}\mathrm{Min}(K, u_\tau)$ (resp. $2\langle v, u_\tau \rangle - l - l\mathrm{Min}(K, u_\tau)$) along $\tilde{p}^{-1}(\mathrm{orb}(\tau))$, and is holomorphic on $\tilde{p}^{-1}(T_N)$ for $v \in M$. Hence $\mathbf{e}^v \tilde{p}^*\omega^{\otimes l}/w^l$ is contained in $H^0(W \setminus p^{-1}(y_0), \mathcal{O}_W(lK_W))$

and not contained in $H^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(lK_{\widetilde{W}} + (l-1)F))$, if and only if $v \in lS$. Since the vertices of $S(\mu)$ are rational points, the increasing order of $|lS(\mu) \cap M|$ is equal to $\dim S(\mu)$.

Let $J = \{m_1, \dots, m_k\}$, $P_i = \mathbf{e}^{2m_i}P$, w_i be as in Section 2. Recall that $w_i^2 = P_i$. $\mathbf{e}^v w_i \tilde{p}^* \omega^{\otimes l}/w^l$ has zeros of order

$$\langle v + m_i, u_\tau \rangle - l - \frac{1}{2}(l-1)\mathrm{Min}(K, u_\tau) \quad (\text{resp. } 2\langle v + m_i, u_\tau \rangle - l - (l-1)\mathrm{Min}(K, u_\tau))$$

along $\tilde{p}^{-1}(\mathrm{orb}(\tau))$, if $\mathrm{Min}(K, u_\tau)$ is even (resp. odd). If $l = 1$ or $\dim S(\mu) < r$, then

$$\{v \in M_{\mathbf{R}} \mid \langle v, u_\tau \rangle \geq l + \frac{1}{2}(l-1)\mathrm{Min}(K, u_\tau) \text{ for } {}^\forall \tau \in \sigma(1),\ \langle v, u_\mu \rangle \leq \frac{1}{2}(l-1)\mathrm{Min}(K, u_\mu)\} = \emptyset$$

and hence there does not exist an element v in M such that $\mathbf{e}^v w_i \tilde{p}^* \omega^{\otimes l}/w^l$ is contained in $H^0(W \setminus p^{-1}(y_0), \mathcal{O}_W(lK_W))$ and has poles of order greater than or equal to l along $\tilde{p}^{-1}(\mathrm{orb}(\mu))$.

Let $lS \cap M = \{v_1, \dots, v_s\}$. Then $[\mathbf{e}^{v_1}\tilde{p}^*\omega^{\otimes l}/w^l]$, …, $[\mathbf{e}^{v_s}\tilde{p}^*\omega^{\otimes l}/w^l]$ span

$$H^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(lK_{\widetilde{W}} + lF))/H^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(lK_{\widetilde{W}} + (l-1)F))$$

and linearly independent. □

If σ satisfies the condition in Proposition 11, K satisfies the conditions (iv), (v) in Theorem 10 and there exists a 1-dimensional cone in Σ with the equality in (v), then $S = \{\frac{1}{2}m_0\}$ and hence $\kappa_\delta(W, p^{-1}(y_0)) = 0$. All examples in Section 3 satisfy these conditions.

Example 7. Let σ be the cone spanned by $\pm 2\mathbf{e}_1 + \mathbf{e}_r$, $\pm\mathbf{e}_2 + \mathbf{e}_r$, …, $\pm\mathbf{e}_{r-1} + \mathbf{e}_r$, where $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r\}$ is a basis of N. Let

$$\Sigma = \{\text{faces of } \sigma(\mathbf{e}_r, \pm\mathbf{e}_1 + \mathbf{e}_r, \dots, \pm\mathbf{e}_{r-1} + \mathbf{e}_r), \sigma(2\epsilon\mathbf{e}_1 + \mathbf{e}_r, \epsilon\mathbf{e}_1 + \mathbf{e}_r, \pm\mathbf{e}_2 + \mathbf{e}_r, \dots, \pm\mathbf{e}_{r-1} + \mathbf{e}_r) \mid \epsilon = \pm 1\},$$

where $\sigma(u_1, u_2, \dots, u_r)$ is the cone spanned by u_1, u_2, …, u_r. Then σ and Σ satisfy the condition of Proposition 11. Let k be an integer with $2 \leq k \leq r-1$ and let

$$K = \{\pm 2\mathbf{e}_2^* \pm \cdots \pm 2\mathbf{e}_{k-1}^* \pm 4\mathbf{e}_k^* \pm \cdots \pm 4\mathbf{e}_{r-1}^* + 4\mathbf{e}_r^*, \pm 2\mathbf{e}_1^* \pm 4\mathbf{e}_2^* \pm \cdots \pm 4\mathbf{e}_{k-1}^* \pm 6\mathbf{e}_k^* \pm \cdots \pm 6\mathbf{e}_{r-1}^* + 6\mathbf{e}_r^*\},$$

where $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \dots, \mathbf{e}_r^*\}$ is the basis of M dual to $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r\}$. Then K satisfies the conditions (i), (ii) in Theorem 3, $\mathrm{Min}(K, \pm 2\mathbf{e}_1 + \mathbf{e}_r) = \mathrm{Min}(K, \pm\mathbf{e}_i + \mathbf{e}_r) = 2$ for $2 \leq i \leq k-1$, $\mathrm{Min}(K, \pm\mathbf{e}_i + \mathbf{e}_r) = 0$ for $k \leq i \leq r-1$ and $\mathrm{Min}(K, \mathbf{e}_r) = \mathrm{Min}(K, \pm\mathbf{e}_1 + \mathbf{e}_r) = 4$. Hence

$$S(\mathbf{R}_{\geq 0}\mathbf{e}_r) = S(\mathbf{R}_{\geq 0}(\pm\mathbf{e}_1 + \mathbf{e}_r)) = \{x_k\mathbf{e}_k^* + \cdots + x_{r-1}\mathbf{e}_{r-1}^* + 2\mathbf{e}_r^* \mid -1 \leq x_i \leq 1 \text{ for } k \leq i \leq r-1\}.$$

Example 8. Let $r = 3$ and let

$$\sigma = \mathbf{R}_{\geq 0}{}^t(2,0,1) + \mathbf{R}_{\geq 0}{}^t(2,1,1) + \mathbf{R}_{\geq 0}{}^t(0,2,1) + \mathbf{R}_{\geq 0}{}^t(-2,1,1) + \mathbf{R}_{\geq 0}{}^t(-2,0,1) + \mathbf{R}_{\geq 0}{}^t(0,-1,1).$$

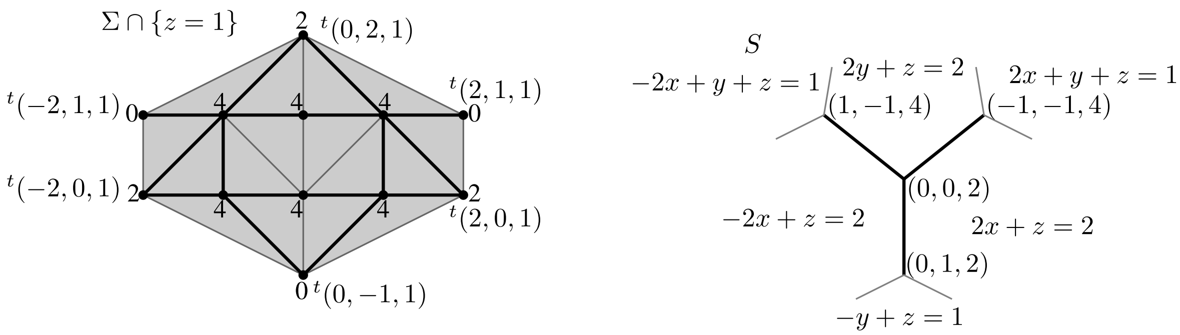


Figure 6

Let Σ be as in Figure 6 and let

$$K=\{(0,0,4),(0,4,4),(0,-2,6),(\pm 4,-6,14),(\pm 4,-2,10),(\pm 2,0,6),(\pm 2,6,6)\}.$$

Then K satisfies the conditions (i), (ii) in Theorem 3 and $S(\mathbf{R}_{\geq 0}{}^t(\pm 1,1,1)) = \overline{(0,0,2)(\mp 1,-1,4)}$, $S(\mathbf{R}_{\geq 0}{}^t(0,1,1)) = \{(0,0,2)\}$, $S(\mathbf{R}_{\geq 0}{}^t(\pm 1,0,1)) = S(\mathbf{R}_{\geq 0}{}^t(0,0,1)) = \overline{(0,0,2)(0,1,2)}$. (See Figure 6. Numbers adjacent vertices denote $\mathrm{Min}(K,u_\tau)$ for the cone τ corresponding the vertices. $|K(\tau)|>1$ for the cones τ corresponding to the vertices and edges drawn by thick lines.) Hence $\kappa_\delta(W,p^{-1}(y_0))=1$.

Let $\widehat{\Sigma}=\{$ faces of $\sigma_i \mid i=1,2,3,4\}$, where

$$\begin{aligned}
\sigma_1 &= \mathbf{R}_{\geq 0}{}^t(2,0,1)+\mathbf{R}_{\geq 0}{}^t(-2,0,1)+\mathbf{R}_{\geq 0}{}^t(0,2,1),\\
\sigma_2 &= \mathbf{R}_{\geq 0}{}^t(2,0,1)+\mathbf{R}_{\geq 0}{}^t(-2,0,1)+\mathbf{R}_{\geq 0}{}^t(0,-1,1),\\
\sigma_3 &= \mathbf{R}_{\geq 0}{}^t(2,0,1)+\mathbf{R}_{\geq 0}{}^t(2,1,1)+\mathbf{R}_{\geq 0}{}^t(0,2,1),\\
\sigma_4 &= \mathbf{R}_{\geq 0}{}^t(-2,0,1)+\mathbf{R}_{\geq 0}{}^t(-2,1,1)+\mathbf{R}_{\geq 0}{}^t(0,2,1).
\end{aligned}$$

Then $|\widehat{\Sigma}|=\sigma$ and Σ is a subdivision of $\widehat{\Sigma}$. Let $\widehat{Y}=T_N\mathrm{emb}(\widehat{\Sigma})$ and let $\widehat{W}$ be the normalization of the hypersurface of $\widehat{Y}\times\mathbf{C}$ defined by $P-w^2=0$. Let $\hat{p}:\widehat{W}\to\widehat{Y}$, $\lambda_1:\widetilde{W}\to\widehat{W}$ and $\lambda_2:\widehat{W}\to W$ be the natural projections. $\widehat{W}$ has only log canonical singularities, by Theorem 10 ($m_0=(0,0,4)$, $(0,2,4)$, $(-2,-2,8)$ and $(2,-2,8)$ for σ_1, σ_2, σ_3 and σ_4, respectively). Let $C_1^{\pm} = \mathrm{orb}(\mathbf{R}_{\geq 0}{}^t(\pm 2,0,1)+\mathbf{R}_{\geq 0}{}^t(0,2,1))$, $C_2=\mathrm{orb}(\mathbf{R}_{\geq 0}{}^t(2,0,1)+\mathbf{R}_{\geq 0}{}^t(-2,0,1))$ in $\widehat{Y}$. Then the fibers of λ_1 over generic points on $\hat{p}^{-1}(C_1^{\pm})$ are elliptic curves and those over $\hat{p}^{-1}(C_2)$ are cycles of 4 rational curves, because fibers of π_1 over generic points of $C_1^{\pm}$ intersect B at 4 points and those over C_2 are chains of 3 rational curves both sides of which intersect B at 2 points, where $\pi_1:\widetilde{Y}\to\widehat{Y}$ is the natural projection.

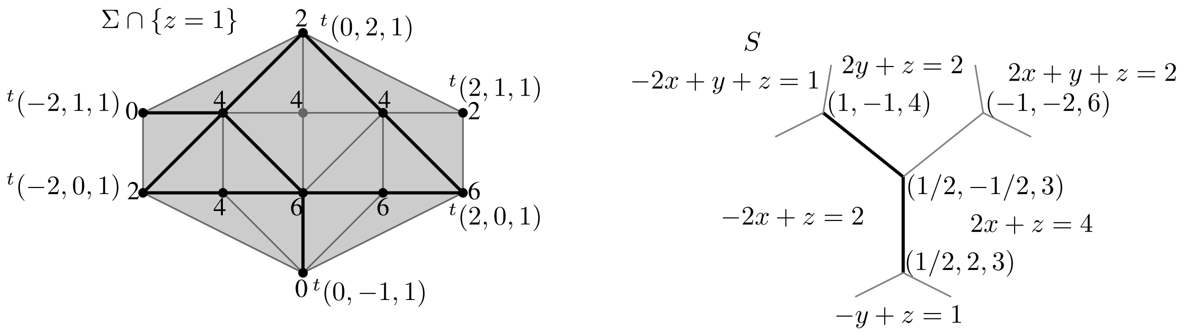


Figure 7

Example 9. Let σ and Σ be as in the above example and let

$$K=\{(0,-2,6),(0,6,6),(2,0,6),(2,6,6),(4,-6,14),(4,-2,10),(-2,-4,10)\}.$$

Then K satisfies the conditions (i), (ii) in Theorem 3 and $S(\mathbf{R}_{\geq 0}{}^t(0,0,1)) = \overline{(1/2,-1/2,3)(1/2,2,3)}$, $S(\mathbf{R}_{\geq 0}{}^t(-1,1,1)) = \overline{(1/2,-1/2,3)(1,-1,4)}$, $S(\mathbf{R}_{\geq 0}{}^t(\pm 1,0,1)) = S(\mathbf{R}_{\geq 0}{}^t(0,1,1)) = S(\mathbf{R}_{\geq 0}{}^t(1,1,1)) = \emptyset$ (see Figure 7). Let $\widehat{\Sigma}=\{$ faces of $\sigma_i \mid i=0,2,4\}$, where σ_2 and σ_4 are as in the above example and

$$\sigma_0=\mathbf{R}_{\geq 0}{}^t(2,0,1)+\mathbf{R}_{\geq 0}{}^t(2,1,1)+\mathbf{R}_{\geq 0}{}^t(0,2,1)+\mathbf{R}_{\geq 0}{}^t(-2,0,1).$$

Let $\widehat{Y}$, $\widehat{W}$, $\hat{p}$, λ_1, λ_2, C_1^- and C_2 be as in the above example. Let $v_1=(1,-2,4)$, $v_2=(-1,1,-2)$, $v_3=(0,1,-1)$. Then $\{v_1,v_2,v_3\}$ is the basis of M dual to $\{{}^t(-1,1,1),{}^t(-2,1,1),{}^t(0,2,1)\}$. Let $z_i=\mathbf{e}^{v_i}$ for $1\leq i\leq 3$. Then (z_2,z_3) is a global coordinate of $T_0:=\mathrm{orb}(\mathbf{R}_{\geq 0}{}^t(-1,1,1))$,

$$P=z_1^4z_3^2\left(c_{(4,-6,14)}+c_{(0,-2,6)}z_2^4+z_3^4(c_{(4,-2,10)}+c_{(2,0,6)}z_2^2)+z_1Q\right)$$

for a polynomial Q, and the restriction of λ_1 to T_0 is expressed as $(z_2, z_3) \mapsto z_2$. Hence the fibers of λ_1 over generic points of $\hat{p}^{-1}(C_1^-)$ are elliptic curves which are biholomorphic to each other. While, the fibers of $\hat{p}^{-1}(C_2)$ consist of 5 rational curves crossing as Figure 3, because B intersects generic fibers of the restriction of $\pi_1 : \widetilde{Y} \to \widehat{Y}$ to $V(\mathbf{R}_{\geq 0}{}^t(0,0,1))$ at 2 points, and does not intersect those to $V(\mathbf{R}_{\geq 0}{}^t(\pm 1,0,1))$.

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