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Examples of four dimensional cusp singularities

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Abstract. We give some examples of four dimensional cusp singularities which are not of Hilbert modular type. We construct them, using quadratic cones and subgroups of reflection groups.

0. Introduction

In [8], we showed that an r-dimensional cusp singularity $\mathrm{Cusp}(C,\Gamma)$ is obtained from a pair (C,Γ) of an open cone C in $\mathbf{R}^r$ and a subgroup Γ of $\mathrm{GL}(r,\mathbf{Z})$ satisfying the following three conditions, where r is an integer greater than 1.

1. C is strongly convex, i.e., $\overline{xy} \subset C$ for any $x, y \in C$ and $\overline{C} \cap \overline{-C} = \{0\}$.
2. C is Γ-invariant, i.e., $\gamma C = C$ for all $\gamma \in \Gamma$.
3. Γ acts on $D_C := C/\mathbf{R}_{>0}$ properly discontinuously, freely and D_C/Γ is compact.

$\mathrm{Cusp}(C,\Gamma)$ is obtained by adding a point to the quotient of the tube domain $\mathbf{R}^r + \sqrt{-1}C$ under the action of the semidirect product of $\mathbf{Z}^r$ and Γ. In the 2-dimensional case, $\mathrm{Cusp}(C,\Gamma)$ is nothing but a Hilbert modular cusp singularity. Hilbert modular cusp singularities exist in all dimensions greater than 1, where C is the interior of a simplicial cone and D_C/Γ is a real torus. It is also known that there exist other higher dimensional cusp singularities of arithmetic type (see [6] and [7, §3], for instance). We gave in [8] some 3-dimensional explicit examples of (C,Γ) such that D_C/Γ are not real tori. In 1991, Ishida[3] gave explicit 4-dimensional examples. Until quite recently no other 4-dimensional explicit examples seem to be found. On the other hand, Vinberg[10] gave a way to obtain a subgroup Γ of $\mathrm{GL}(r,\mathbf{R})$ acting properly discontinuously on a strongly convex open cone C in $\mathbf{R}^r$. Here Γ is generated by reflections with respect to the hyperplanes containing the $(r-1)$-dimensional faces of a polyhedral cone satisfying certain conditions. Moreover, he gave a simple necessary and sufficient condition for the cone C to be quadratic, i.e., defined by a quadratic polynomial. In this paper, using the results in [10], we give some explicit examples of 4-dimensional pairs (C,Γ) such that Γ are subgroups of reflection groups.

In Section 1, we show that for any open strongly convex cone C in $\mathbf{R}^r$, any subgroup of $\mathrm{GL}(r,\mathbf{Z})$ preserving C, acts on D_C properly discontinuously. In Section 2, we show that if a quadratic polynomial P defines a cone C in $\mathbf{R}^r$ and there exists a subgroup Γ of $\mathrm{GL}(r,\mathbf{Z})$ satisfying the above conditions, then all coefficients of P may be assumed to be integers and $P(x) \neq 0$ for any point x in $\mathbf{Z}^r \backslash \{0\}$. In Section 3, we show that if a quadratic cone C contains a rational polyhedral cone satisfying certain conditions, then there exists a reflection group Γ contained in $\mathrm{GL}(r,\mathbf{Z})$ and acting on C with compact D_C/Γ. In Section 4, we study the structure of exceptional sets of resolutions of $\mathrm{Cusp}(C,\Gamma)$ for

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pairs (C, Γ) such that Γ is a subgroup of a reflection group. Finally, we give three 4-dimensional examples of pairs (C, Γ) with quadratic C, and one with non-quadratic C and a resolution of $\mathrm{Cusp}(C, \Gamma)$ whose exceptional set consists of 4 irreducible components.

1. Groups acting on cones

Let N be a free $\mathbf{Z}$-module of rank $r > 1$, let $M = \mathrm{Hom}(N, \mathbf{Z})$ and let $\langle\ ,\ \rangle : M \times N \longrightarrow \mathbf{Z}$ be the natural pairing. For an open cone C in $N_{\mathbf{R}} = N \otimes \mathbf{R}$, let $D_C = C/\mathbf{R}_{>0}$ and let $p_C : C \longrightarrow D_C$ be the natural projection.

DEFINITION. $\Gamma_C = \{\gamma \in \mathrm{GL}(N) \mid \gamma C = C\}$ for an open cone C in $N_{\mathbf{R}}$.

Let $C^* = \{x \in M_{\mathbf{R}} \mid \langle x, y\rangle > 0 \text{ for } y \in \overline{C} \setminus \{0\}\}$. If C is an open strongly convex cone in $N_{\mathbf{R}}$, then $\Gamma_{C^*} = \{{}^t\gamma \mid \gamma \in \Gamma_C\}$, where ${}^t\gamma$ is the element in $\mathrm{GL}(M)$ satisfying $\langle {}^t\gamma x, y\rangle = \langle x, \gamma y\rangle$ for any elements x and y in M and N, respectively.

THEOREM 1. *If C is an open strongly convex cone in $N_{\mathbf{R}}$, then Γ_C acts on D_C properly discontinuously, i.e., $\{\gamma \in \Gamma \mid \gamma S \cap S \neq \varnothing\}$ is finite for every compact subset S of D_C.*

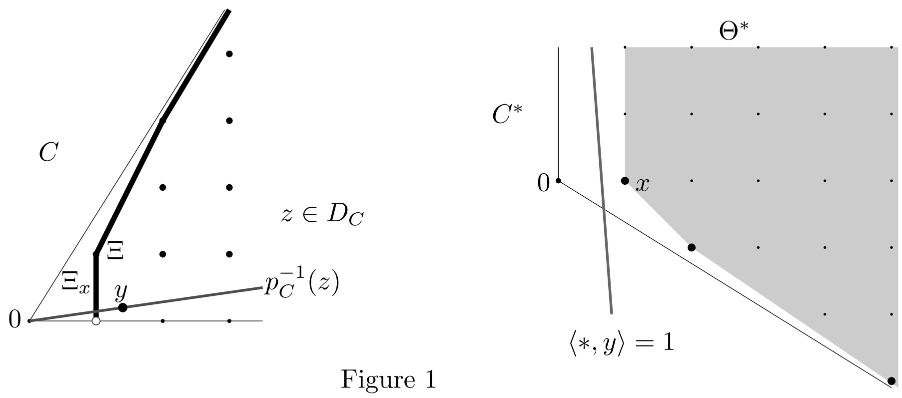


Figure 1

PROOF. Let Θ^* be the convex hull of $C^* \cap M$ and let Ξ be the boundary of $\{y \in C \mid \langle x, y\rangle \geq 1 \text{ for } x \in \Theta^*\}$. Then the restriction $p_{C|\Xi} : \Xi \longrightarrow D_C$ of p_C to Ξ is a homeomorphism (see Figure 1). Let $\Xi_x = \{y \in \Xi \mid \langle x, y\rangle = 1\}$ for each element x in $C^* \cap M$. Then Ξ_x is closed in Ξ. Let L be the set of vertices on Θ^*. Then L is contained in M and $\Xi = \bigcup_{x \in L} \Xi_x$. For any point y in Ξ, $\{x \in L \mid y \in \Xi_x\} \subset \{x \in C^* \cap M \mid \langle x, y\rangle = 1\}$ is finite.

Let S be a compact subset of D_C. Then $L_0 = \{x \in L \mid S \cap p_C(\Xi_x) \neq \varnothing\}$ is finite. If $\gamma S \cap S \neq \varnothing$ for an element γ in Γ_C, then there exist elements x_1, x_2 in L_0 with ${}^t\gamma x_1 = x_2$. On the other hand, $K = \{y \in C \cap N \mid \langle x_1, y\rangle = c\}$ contains linearly independent r elements for a positive integer c. Then $\{\gamma \in \Gamma_C \mid {}^t\gamma x_1 = x_1\} \subset \{\gamma \in \Gamma_C \mid \gamma K = K\}$ is a finite set. Hence $\{\gamma \in \Gamma_C \mid {}^t\gamma x_1 = x_2\}$ is also finite for any elements x_1, x_2 in L_0. Therefore, $\{\gamma \in \Gamma_C \mid \gamma S \cap S \neq \varnothing\}$ is finite. □

For an open strongly convex cone C with compact D_C/Γ_C, there exists a normal subgroup Γ of Γ_C with a finite index acting on D_C freely. For example, we obtain such a group as the intersection with the kernel of $\mathrm{SL}(N) \to \mathrm{SL}(N/nN)$ for a suitable positive integer n.

2. Quadratic cones

We fix a coordinate $(X_1, X_2, \ldots, X_r)$ of N throughout the rest of this paper. For a homogeneous polynomial $P(X_1, X_2, \ldots, X_r)$ of r variables, we denote by C_P the open cone defined by

$$\{(x_1, x_2, \ldots, x_r) \in N_{\mathbf{R}} \mid P(x_1, x_2, \ldots, x_r) > 0\}.$$

Definition. We call a cone C in $N_{\mathbf{R}}$ quadratic, if there exists a homogeneous quadratic polynomial $P(X_1, X_2 \ldots, X_r)$ such that C is a connected component of C_P.

If a quadratic cone C defined by a polynomial P is strongly convex, then the signature of P is $(1, r-1)$ and $C \cup (-C) = C_P$.

Theorem 2. *Let C be a quadratic strongly convex cone in $N_{\mathbf{R}}$ defined by a polynomial P. If D_C/Γ_C is compact, then there exists a positive real number c such that all coefficients of cP are integers and P has no isotropic elements in N, i.e., $P(x) \neq 0$ for all x in $N \setminus \{0\}$.*

Proof. First, we show that there exists a finite set K contained in $C \cap N$ such that the convex hull of $p_C(\Gamma_C K)$ is equal to D_C. Let Ξ be the boundary of the convex hull of $C \cap N$ and let $J = \Xi \cap N$. Then the convex hull of $p_C(J)$ is equal to D_C. On the other hand, J/Γ_C is finite, because D_C/Γ_C is compact. Hence there exists a finite set K such that $\Gamma_C K = J$.

Let x be an element in K. We may assume that $P(x) = 1$, multiplying P by a positive number. Then $P(\gamma x) = 1$ for any element γ in Γ_C. Hence all coefficients of P are rational, by the following lemma.

Lemma. There exist $m = \frac{r(r+1)}{2}$ elements γ_1, γ_2, …, γ_m in Γ_C and an element x in K such that $f(\gamma_1 x)$, $f(\gamma_2 x)$, …, $f(\gamma_m x)$ are linearly independent, where $f : N \longrightarrow \mathbf{Z}^m$ is the map sending $(x_1, x_2, \ldots, x_r)$ to $(x_1^2, \ldots, x_r^2, x_1 x_2, \ldots, x_{r-1} x_r)$.

Proof. Suppose that $f(\gamma_1 x)$, $f(\gamma_2 x)$, …, $f(\gamma_m x)$ are linearly dependent for any element x in K and any m elements γ_1, γ_2, …, γ_m in Γ_C. Then $f(\Gamma_C x)$ is contained in an $(m-1)$-dimensional linear subspace of $\mathbf{R}^m$. It implies that there exists a homogeneous quadratic polynomial $Q_x(x_1, x_2, \ldots, x_r)$ such that $Q_x(\gamma x) = 0$ for all γ in Γ_C. Since K is finite, there exists a point x_0 on $\partial C \setminus \{0\}$ such that $Q_x(x_0) \neq 0$ for all x in K. Then there exists a non-zero element y_0 in $M_{\mathbf{R}}$ such that $\langle y_0, x_0 \rangle < 0$ and that $\langle y_0, \gamma x \rangle > 0$ for all x in K and for all γ in Γ_C, because there exists a hyperplane H with $H \cap \partial C = \mathbf{R}_{\geq 0} x_0$. Hence D_C is not equal to the convex hull of $p_C(\Gamma_C K)$, a contradiction.

Next, suppose that $P(y_0) = 0$ for an element y_0 in $N \setminus \{0\}$. We may assume that y_0 is primitive and that $y_0 \in \partial C$. Let x_0 be a vertex on the boundary of the convex hull of

$\{x \in C^* \cap M \mid \langle x, y_0 \rangle = 1\}$, which is not empty. Then $x_0 \in M$ and $y_0 \in \overline{\Theta_{x_0}}$, where

$$\Theta_{x_0} = \{y \in C \mid \langle x_0, y \rangle = 1, \langle x, y \rangle \geq 1 \text{ for } x \in C^* \cap M\}.$$

Since $\overline{\Theta_{x_0}}$ is compact, $\Theta_{x_0} \cap N$ is a finite set. Hence $\Gamma_0 = \{\gamma \in \Gamma_C \mid \gamma\Theta_{x_0} = \Theta_{x_0}\}$ is a finite group. Therefore, $p_C(\Theta_{x_0})/\Gamma_0$ is not compact. However, $p_C(\Theta_{x_0})$ is closed in D_C. It implies that D_C/Γ_C is not compact. □

In the 2-dimensional case, the converse of the above theorem holds, because $C = \mathbf{R}_{\geq 0}v_1 + \mathbf{R}_{\geq 0}v_2$ for two eigenvectors v_1 and v_2 in $N_\mathbf{R} \setminus N_\mathbf{Q}$ of an element in $\mathrm{SL}(N)$.

Proposition 3. *An open strongly convex cone C in $N_\mathbf{R}$ with compact D_C/Γ_C, is quadratic, if and only if there exists a homomorphism $f : N \to M$ such that $f_\mathbf{R}(C) = C^*$ and that $f \circ \gamma = {}^t\gamma^{-1} \circ f$ for any element γ in Γ_C.*

Proof. Assume that C is quadratic, i.e., there exists a regular symmetric matrix A of index $(1, r-1)$ such that C is a connected component of $\{x \in N_\mathbf{R} \mid {}^t xAx > 0\}$. We may assume that all entries of A are integers, by Theorem 2. Let $f : N \to M$ be the homomorphism satisfying $\langle f(y), x \rangle = {}^t yAx$. Since the index of A is $(1, r-1)$,

$$\{y \in N_\mathbf{R} \mid {}^t yAx > 0 \text{ for } x \in \overline{C} \setminus \{0\}\} = C.$$

Therefore, $f_\mathbf{R}(C) = C^*$. Let γ be any element in Γ_C. Then ${}^t\gamma A\gamma = A$. Hence

$$\langle f(\gamma y), x \rangle = {}^t(\gamma y)Ax = {}^t y {}^t\gamma Ax = {}^t yA\gamma^{-1}x = \langle f(y), \gamma^{-1}x \rangle = \langle {}^t\gamma^{-1} f(y), x \rangle.$$

Therefore, $f \circ \gamma = {}^t\gamma^{-1} \circ f$.

Conversely, assume that there exists a homomorphism $f : N \to M$ as in the proposition. We define a symmetric bilinear form on $N_\mathbf{R}$ by $x \cdot y = \langle f_\mathbf{R}(x), y \rangle + \langle f_\mathbf{R}(y), x \rangle$. Then there exists a symmetric and integer matrix A with $x \cdot y = {}^t xAy$. For any element γ in Γ_C, $\gamma x \cdot \gamma y = x \cdot y$, because $\langle f_\mathbf{R}(\gamma x), \gamma y \rangle = \langle {}^t\gamma^{-1} f_\mathbf{R}(x), \gamma y \rangle = \langle f_\mathbf{R}(x), y \rangle$. Since $f_\mathbf{R}(C) = C^*$, $x \cdot y > 0$ for any points x and y in C. Hence $x \cdot x \geq 0$ for any point x on ∂C, because the function $N_\mathbf{R} \ni x \mapsto x \cdot x \in \mathbf{R}$ is continuous. Let Θ be the convex hull of $C \cap N$. Since $\partial\Theta/\Gamma_C$ is compact, $\{x \cdot x \mid x \in \partial\Theta\}$ has the maximal value d. Let $S_d = \{x \in N_\mathbf{R} \mid x \cdot x = d\}$. Then $S_d \cap C \subset \Theta$. Since Θ is closed and $\Theta \cap \partial C = \varnothing$, $S_d \cap \partial C = \varnothing$. Hence $x \cdot x = 0$ for any point x on ∂C. Therefore, C is a connected component of $\{x \in N_\mathbf{R} \mid x \cdot x > 0\}$. □

The above proposition can be applied to decide whether the cone C is quadratic for a pair (C, Γ) satisfying the conditions 1, 2 and 3 in Introduction. We give an example. Let $r = 3$. Let S be the surface and Δ be its triangulation obtained from the hexagon in Fugure 2, identifying the edges $\overline{v_1v_2}$, $\overline{v_3v_4}$ and $\overline{v_5v_6}$ with $\overline{v_2v_3}$, $\overline{v_4v_5}$ and $\overline{v_6v_1}$, respectively. Then $\chi(S) = -1$ and the double $\mathbf{Z}$-weight on Δ as in Figure 2 satisfies the monodromy condition and the convexity condition (see [8, Definitions 1.3 and 1.5]). Hence we obtain a map σ : {all vertices of $\widetilde{\Delta}$} $\to N$ and a homomorphism $\rho : \pi_1(S) \to \mathrm{GL}(N)$ such that $\sigma(\gamma v) = \rho(\gamma)\sigma(v)$ for all vertices v of $\widetilde{\Delta}$ and all elements γ in $\pi_1(S)$ by [8], where $\widetilde{\Delta}$ is the pull-back of Δ under the universal covering $\varpi : \widetilde{S} \to S$. Let $C = \mathbf{R}_{>0}\Theta$, where Θ is the convex hull of the image of σ, and let $\Gamma = \rho(\pi_1(S))$.

Then the pair (C,Γ) satisfies the conditions 1, 2 and 3 in Introduction. There exist vertices $\tilde{v}_1$, $\tilde{v}_2$, …, $\tilde{v}_6$ of $\widetilde{\Delta}$ with $\varpi(\tilde{v}_i) = v_i$ such that $\overline{\tilde{v}_1\tilde{v}_2\tilde{v}_3}$, $\overline{\tilde{v}_3\tilde{v}_4\tilde{v}_5}$, $\overline{\tilde{v}_5\tilde{v}_6\tilde{v}_1}$ and $\overline{\tilde{v}_1\tilde{v}_3\tilde{v}_5}$ are triangles of $\widetilde{\Delta}$. Here we may assume that $\sigma(\tilde{v}_1) = \mathbf{e}_1$, $\sigma(\tilde{v}_3) = \mathbf{e}_2$ and $\sigma(\tilde{v}_5) = \mathbf{e}_3$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of N. Let $\mathbf{f}_i = \sigma(\tilde{v}_{2i}) (= 2\mathbf{e}_i + 2\mathbf{e}_{i+1} - \mathbf{e}_{i+2})$ for each i in $\mathbf{Z}/3\mathbf{Z}$. Let $\Sigma = \{\gamma\tau \mid \gamma \in \Gamma, \tau \prec \mu_i, i = 0,1,2,3\}$, where $\mu_0 = \mathbf{R}_{\geq 0}\mathbf{e}_1 + \mathbf{R}_{\geq 0}\mathbf{e}_2 + \mathbf{R}_{\geq 0}\mathbf{e}_3$ and $\mu_i = \mathbf{R}_{\geq 0}\mathbf{e}_i + \mathbf{R}_{\geq 0}\mathbf{e}_{i+1} + \mathbf{R}_{\geq 0}\mathbf{f}_i$ for $i = 1,2,3$. Then Σ is a non-singular fan with $|\Sigma| \setminus \{0\} = C$ and Γ acts on the set of 1-dimensional cones in Σ transitively, because Δ has only one vertex. Hence we have a resolution of $\mathrm{Cusp}(C,\Gamma)$ whose exceptional set is irreducible.

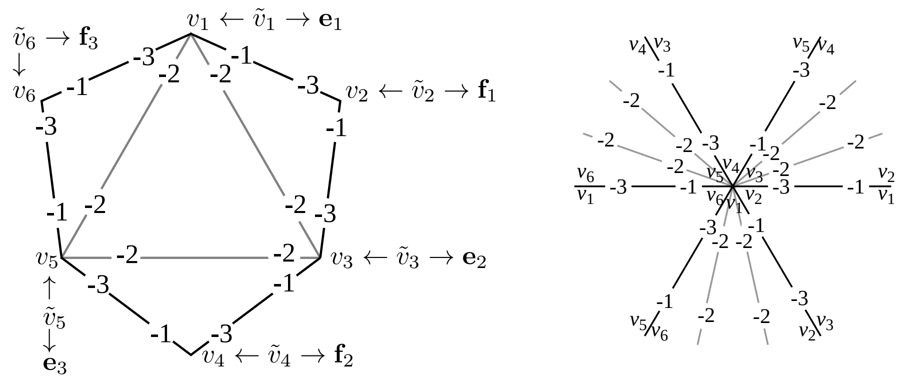


Figure 2

Proposition 4. *The above cone C is not quadratic.*

Proof. Let γ_i be the elements in $\mathrm{GL}(N)$ sending $\mathbf{e}_i$, $\mathbf{f}_i$ and $\mathbf{e}_{i+1}$ to $\mathbf{f}_i$, $\mathbf{e}_{i+1}$ and $\mathbf{f}_i + 3\mathbf{e}_{i+1} - \mathbf{e}_i$, respectively for all i in $\mathbf{Z}/3\mathbf{Z}$. Then γ_i are in Γ_C. We easily see that also is in Γ_C the element sending $\mathbf{e}_i$ to $\mathbf{e}_{i+1}$, which we denote by δ. Let $\mathbf{e}_0 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{e}_0^* = \mathbf{e}_1^* + \mathbf{e}_2^* + \mathbf{e}_3^*$, where $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$ is the basis of M dual to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then $\delta\mathbf{e}_0 = \mathbf{e}_0$ and ${}^t\delta\mathbf{e}_0^* = \mathbf{e}_0^*$. Suppose that there exists an injective homomorphism $f : N \to M$ satisfying $f \circ \gamma = {}^t\gamma^{-1} \circ f$ for any element γ in Γ_C. Then $f(\mathbf{e}_0) = c\mathbf{e}_0^*$ for a non-zero integer c, because any fixed point of ${}^t\delta^{-1}$ is on $\mathbf{R}\mathbf{e}_0^*$. We see by an easy calculation that $\gamma_i\mathbf{e}_0 = 9\mathbf{e}_i + 20\mathbf{e}_{i+1} - 6\mathbf{e}_{i+2}$ and ${}^t\gamma_i^{-1}\mathbf{e}_0^* = 9\mathbf{e}_i^* + 3\mathbf{e}_{i+1}^* + 23\mathbf{e}_{i+2}^*$. Hence $\gamma_1\mathbf{e}_0 + \gamma_2\mathbf{e}_0 + \gamma_3\mathbf{e}_0 = 23\mathbf{e}_0$ and ${}^t\gamma_1^{-1}\mathbf{e}_0^* + {}^t\gamma_2^{-1}\mathbf{e}_0^* + {}^t\gamma_3^{-1}\mathbf{e}_0^* = 35\mathbf{e}_0^*$. It imlpies $c = 0$. Hence C is not quadratic, by Proposition 3. □

3. Reflections

Let P be a quadratic homogeneous polynomial of r variables with the signature $(1, r-1)$, and let C be a connected component of C_P. Then C is strongly convex and $C_P = C \cup (-C)$. We assume that all coefficients of P are integers with no common divisors greater than 1, throughout this section. Let $B_P : N \times N \to \mathbf{Z}$ be the symmetric bilinear form with $B_P(x,x) = 2P(x)$.

Definition. $x \cdot y = B_P(x,y)$ for elements $x, y \in N_{\mathbf{R}}$.

We easily see that $\gamma x \cdot \gamma y = x \cdot y$ for any element γ in Γ_C. For an element v in $N_{\mathbf{R}}$

with $v \cdot v \neq 0$, we define a linear transformation γ_v and a hyperplane H_v of $N_{\mathbf{R}}$ as follows:

$$\gamma_v : x \mapsto x - 2\frac{x \cdot v}{v \cdot v}v, \qquad H_v = \{x \in N_{\mathbf{R}} \mid x \cdot v = 0\}.$$

We see by easy calculation that $\gamma_v^2 = \mathrm{id}$, $\gamma_v v = -v$, $\gamma_v x = x$ for any x in H_v and $\gamma_v x \cdot \gamma_v y = x \cdot y$ for any x, y in $N_{\mathbf{R}}$. Hence $\gamma_v C = C$ or $-C$. If $v \cdot v < 0$, then $\gamma_v C = C$, because $C \cap H_v \neq \varnothing$. Hence we have:

PROPOSITION 5. *If v is an element in N with $v \cdot v < 0$ and $2\frac{\mathbf{e}_i \cdot v}{v \cdot v} \in \mathbf{Z}$ for each fundamental vector $\mathbf{e}_i$, then γ_v is in Γ_C.*

Any element v in N with $v \cdot v = -2$ satisfies the assumption of the above proposition. Let $F_\gamma = \{x \in C \mid \gamma x = x\}$ for an element γ in Γ_C.

PROPOSITION 6. *Let γ be an element in Γ_C with $F_\gamma \neq \varnothing$ and $\dim F_\gamma = r - 1$. Then there exists an element v in N with $\gamma = \gamma_v$.*

PROOF. $r - 1$ of the eigenvlues of γ are equal to 1. The other is equal to -1 and $\gamma^2 = 1$, by Theorem 1. Hence there exists a non-zero element v in N with $\gamma v = -v$. For any element x in $N_{\mathbf{R}}$, there exists a real number c_x with $x - \gamma x = c_x v$, because $\gamma(x - \gamma x) = -(x - \gamma x)$. On the other hand, $\gamma x \cdot v = x \cdot \gamma v$, because $\gamma^2 = 1$. Hence $(x - \gamma x) \cdot v = 2x \cdot v$. Therefore, $c_x = 2\frac{x \cdot v}{v \cdot v}$. □

Here we note that an eigenvector h of γ_v corresponding to the eignevalue -1 and the linear function α on $N_{\mathbf{R}}$ with $\alpha(h) = 2$ and vanishing on H_v in [10], are nothing but v and the function $\alpha(x) = 2v \cdot x / v \cdot v$, respectively.

PROPOSITION 7. *Let v and w be elements in N with $v \cdot v < 0$ and $w \cdot w < 0$. If $\frac{v \cdot w}{\sqrt{-v \cdot v}\sqrt{-w \cdot w}} = 0, \frac{1}{2}, \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{3}}{2}$, then $|\gamma_v \gamma_w| = 2, 3, 4$ or 6, respectively, and $\lambda = \{y \in N_{\mathbf{R}} \mid v \cdot y \geq 0, w \cdot y \geq 0\}$ is a fundamental domain of the action of $\langle \gamma_v, \gamma_w \rangle$ on $N_{\mathbf{R}}$.*

PROOF. We may assume that $v \cdot v = w \cdot w = -1$ replacing v and w with $v/\sqrt{-v \cdot v}$ and $w/\sqrt{-w \cdot w}$, respectively. Assume that $v \cdot w = \sqrt{3}/2$. Then $\gamma_v \gamma_w$ sends v and w to $2v + \sqrt{3}w$ and $-\sqrt{3}v - w$, respectively. Hence $|\gamma_v \gamma_w| = 6$. Moreover,

$$\lambda = \mathbf{R}_{\geq 0}(-2v - \sqrt{3}w) + \mathbf{R}_{\geq 0}(-\sqrt{3}v - 2w) + \{y \in N_{\mathbf{R}} \mid v \cdot y = w \cdot y = 0\}.$$

We see by easy calculation that $r - 2 \leq \dim(\gamma\lambda \cap \lambda) \leq r - 1$ for any γ in $\langle \gamma_v, \gamma_w \rangle \setminus \{1\}$. For the other cases, calculation is easier. □

If $\frac{v \cdot w}{\sqrt{-v \cdot v}\sqrt{-w \cdot w}} = -\frac{1}{2}, -\frac{1}{\sqrt{2}}$ or $-\frac{\sqrt{3}}{2}$, then $|\gamma_v \gamma_w| = 3, 4$ or 6, respectively, however, $\dim(\gamma_v \gamma_w \gamma_v \lambda \cap \lambda) = r$. Let σ be an r-dimensional rational polyhedral cone. For each $(r-1)$-dimensional face τ of σ, we denote by $v(\tau)$ the unique primitive element v in N determined by the condition that $v \cdot y = 0$ for all points y in τ and $v \cdot y \geq 0$ for all points y in σ.

THEOREM 8. *If there exists an r-dimensional rational polyhedral cone σ satisfying the following three conditions, then $p_C(\sigma \setminus \{0\})$ is a fundamental domain of the action of Γ on D_C, $\Sigma = \{\gamma\lambda \mid \gamma \in \Gamma, \lambda \prec \sigma\}$ is a fan and $|\Sigma| = C \cup \{0\}$, where $\Gamma = \langle \gamma_{v(\tau)} \mid \tau \prec \sigma, \dim \tau = n-1\rangle$.*

1. $\sigma \setminus \{0\} \subset C$.

2. $v(\tau)\cdot v(\tau) < 0$ and $\gamma_{v(\tau)} \in \Gamma_C$ for any $(r-1)$-dimensional face τ of σ.

3. $\frac{v(\tau)\cdot v(\mu)}{\sqrt{-v(\tau)\cdot v(\tau)}\sqrt{-v(\mu)\cdot v(\mu)}} = 0, \frac{1}{2}, \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{3}}{2}$ for any $(r-1)$-dimensional faces τ and μ of σ with $\dim(\tau \cap \mu) = r-2$.

PROOF. We can define distance $\overline{vw}$ on $S_C = \{v \in C \mid v\cdot v = 1\} \simeq D_C$ by $\cosh \overline{vw} = v \cdot w$ and angle $\angle H_v^C H_w^C$ of two hyperplanes $H_v^C = H_v \cap S_C$ and $H_w^C = H_w \cap S_C$ on S_C by $\cos \angle H_v^C H_w^C = \frac{v\cdot w}{\sqrt{-v\cdot v}\sqrt{-w\cdot w}}$ for $v, w \in N_{\mathbf{R}}$ with $v\cdot v < 0$, $w \cdot w < 0$. Then we may regard D_C as a hyperbolic space and $(p_C)_{\mathbf{R}}(\sigma \setminus \{0\})$ as a Coxeter polyhedron, by the conditions 2, 3 and Proposition 7. Hence we see by [4, Theorem 7.1.3] that the assertions of the theorem hold. □

4. Structure of exceptional sets

We keep the notations and the assumptions in the previous section. Let σ be an r-dimensional rational polyhedral cone satisfying the conditions of Theorem 8. Let $W = T_N\mathrm{emb}(\Sigma)$ be the toric variety associated to the fan Σ in Theorem 8. For a cone $\tau \neq \{0\}$ in Σ, we denote by $V(\tau)$ the closure of $\mathrm{orb}(\tau)$ in W, which is a compact toric variety (see [5, Corollary 1.7]). Let $\mathrm{ord} : T_N \to N_{\mathbf{R}}$ be the homomorphism induced by $-\log|\ | : \mathbf{C}^\times \to \mathbf{R}$. Let $\widetilde{U}$ be the interior of the closure of $\mathrm{ord}^{-1}(C)$ in W and let $\widetilde{X} = W \setminus T_N$. Then $\widetilde{U}$ is an open neighborhood of $\widetilde{X}$. Let Γ_0 be a subgroup of Γ with a finite index acting on D_C freely. Then Γ_0 acts on $\widetilde{U}$ freely. Let $U = \widetilde{U}/\Gamma_0$ and let $X = \widetilde{X}/\Gamma_0$. Then the cusp singularity $\mathrm{Cusp}(C, \Gamma_0)$ is obtained by contracting X to a point in U (see [8]).

Let λ be a face of σ with $1 \le s := \dim\lambda \le r-2$, and let $p_\lambda : N \to N/(\mathbf{R}\lambda \cap N)$ be the natural projection. Let μ_1, μ_2, …, μ_l be the $(r-1)$-dimensional faces of σ with $\lambda \prec \mu_i$ and let $\Gamma_\lambda = \langle \gamma_{v(\mu_i)} \mid i = 1, \ldots, l\rangle$. Then Γ_λ acts on $N/(\mathbf{R}\lambda \cap N)$. Let $\Sigma_\lambda = \{(p_\lambda)_{\mathbf{R}}(\tau) \mid \tau \in \Sigma, \lambda \prec \tau\}$. Then Σ_λ is a Γ_λ-invariant fan in $N/(\mathbf{R}\lambda\cap N)$. Moreover, $V(\lambda) \simeq T_{N/(\mathbf{R}\lambda\cap N)}\mathrm{emb}(\Sigma_\lambda)$, by [5, Corollary 1.7]. Hence $V(\lambda)$ is non-singular, if and only if so is $(p_\lambda)_{\mathbf{R}}(\sigma)$.

Now, assume that $(p_\lambda)_{\mathbf{R}}(\sigma)$ is non-singular, i.e., $(p_\lambda)_{\mathbf{R}}(\sigma) = \mathbf{R}_{\ge 0}w_1 + \mathbf{R}_{\ge 0}w_2 + \cdots + \mathbf{R}_{\ge 0}w_{r-s}$ for a basis $\{w_1, w_2, \ldots, w_{r-s}\}$ of $N/(\mathbf{R}\lambda\cap N)$. Then there exist elements u_1, u_2, …, u_{r-s} in $N \cap \sigma$ with $w_i = p_\lambda(u_i)$. Let $\{u_{r-s+1}, \ldots, u_r\}$ be a basis of $\mathbf{R}\lambda \cap N$. Then $\{u_1, u_2, \ldots, u_r\}$ is a basis of N. Moreover, so is $\{u_1, \ldots, u_{i-1}, \gamma_{v(\mu_i)}u_i, u_{i+1}, \ldots, u_r\}$, because $\gamma_{v(\mu_i)}$ is in $\mathrm{GL}(N)$ and $\gamma_{v(\mu_i)}u_j = u_j$ if $i \neq j$. Hence there exist integers $c_{i,j}$ $(1 \le i \le r-s, 1 \le j \le r)$ with

$$u_i + \gamma_{v(\mu_i)}u_i + c_{i,1}u_1 + \cdots + c_{i,i-1}u_{i-1} + c_{i,i+1}u_{i+1} + \cdots + c_{i,r}u_r = 0.$$

Therefore,

$$w_i + \gamma_{v(\mu_i)}w_i + c_{i,1}w_1 + \cdots + c_{i,i-1}w_{i-1} + c_{i,i+1}w_{i+1} + \cdots + c_{i,r-s}w_{r-s} = 0.$$

These numbers $c_{i,j}$ determine the structure of $V(\lambda)$. Especially, when $s = r-3$, they are nothing but double **Z**-weights in [5, 1.7]. We easily see that $c_{i,j} \leq 0$. Moreover, $|\gamma_{v(\mu_i)}\gamma_{v(\mu_j)}| = +\infty$, if $c_{i,j} \leq -2$ and $c_{j,i} \leq -2$, $c_{i,j} = -1$ and $c_{j,i} \leq -4$ or $c_{i,j} = 0$ and $c_{j,i} \neq 0$. Hence if $v(\mu_i)\cdot v(\mu_j)/\left(\sqrt{-v(\mu_i)\cdot v(\mu_i)}\sqrt{-v(\mu_j)\cdot v(\mu_j)}\right) = 0, \frac{1}{2}, \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{3}}{2}$, then $\{c_{i,j}, c_{j,i}\} = \{0\}, \{-1\}, \{-1,-2\}$ or $\{-1,-3\}$, respectively, by Proposition 7.

We explain some examples of $V(\lambda)$ for the convenience of the next section. First, we consider the case $s = r-2$ and $(p_\lambda)_{\mathbf{R}}(\sigma)$ is non-singular. If $c_{1,2} = c_{2,1} = 0$, then $V(\lambda) \simeq \mathbf{P}^1 \times \mathbf{P}^1$. If $c_{1,2} = c_{2,1} = -1$, then $V(\lambda) \simeq S_6$. If $c_{1,2} = -1$ and $c_{2,1} = -2$ (resp. -3), then $V(\lambda) \simeq S_8$ (resp. S_{12}). Here S_i are toric surfaces obtained from Coxeter groups as follows (see [2, 5.1] for the definition of Coxeter group). For each $i = 6, 8, 12$, let G_i be a subgroup of $\mathrm{GL}(2,\mathbf{Z})$ generated by two elements g_1 and $g_{2,i}$ defined by

$$g_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad g_{2,6} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad g_{2,8} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad g_{2,12} = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}.$$

Then G_i are Coxeter groups with $|G_i| = i$. Let $\Lambda_i = \{\text{faces of } g\mathbf{R}^2_{\geq 0} \mid g \in G_i\}$. Then Λ_i is a non-singular fan for each i. Let $S_i = T_{\mathbf{Z}^2}\mathrm{emb}(\Lambda_i)$ be the compact toric surface associated to the fan Λ_i. Then the complement of the algebraic torus in S_6, is a cycle of 6 rational curves with the self-intersection numbers all equal to -1. The complement of the algebraic torus in S_8 (resp. S_{12}), is a cycle of 8 (resp. 12) rational curves with the self-intersection numbers repeating -1, -2 (resp. -1, -3).

Next, we consider the case $s = r-3$ and assume that $(p_\lambda)_{\mathbf{R}}(\sigma)$ is non-singular except the case (7). We denote by V_i the toric variety $V(\lambda)$ in (i), which appears in the following sections as an irreducible component of the exceptional set of a resolution of 4-dimensional cusp singularities.

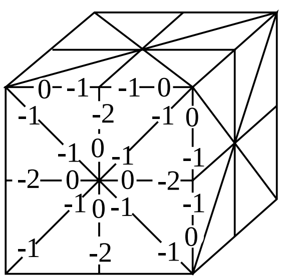


Figure 3

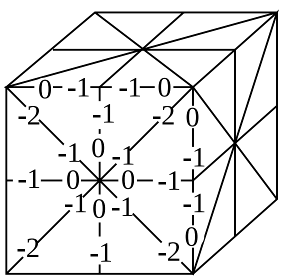


Figure 4

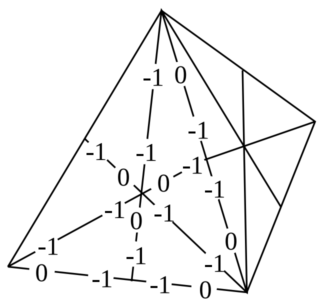


Figure 5

(1a) If $c_{1,2} = c_{2,1} = 0$, $c_{1,3} = c_{3,1} = c_{3,2} = -1$, $c_{2,3} = -2$, then the complement of the algebraic torus in V_{1a}, consists of 26 toric surfaces 6, 8 and 12 of which are biholomorphic to S_8, S_6 and $\mathbf{P}^1 \times \mathbf{P}^1$, respectively (see Figure 3). The self-intersection numbers $(E_{|V})^2$ in irreducible components $V \simeq S_8$ of rational curves $E = V \cdot W$, are equal to -2 and -1, if $W \simeq \mathbf{P}^1 \times \mathbf{P}^1$ and S_6, respectively.

(1b) If $c_{1,2} = c_{2,1} = 0$, $c_{1,3} = c_{3,1} = c_{2,3} = -1$, $c_{3,2} = -2$, then the complement of the algebraic torus in V_{1b}, consists of 26 toric surfaces 6, 8 and 12 of which are biholomorphic to S_8, S_6 and $\mathbf{P}^1 \times \mathbf{P}^1$, respectively (see Figure 4). The self-intersection numbers $(E_{|V})^2$

in irreducible components $V \simeq S_8$ of rational curves $E = V \cdot W$, are equal to -1 and -2, if $W \simeq \mathbf{P}^1 \times \mathbf{P}^1$ and S_6, respectively.

(2) If $c_{1,2} = c_{2,1} = 0$, $c_{1,3} = c_{3,1} = c_{3,2} = c_{2,3} = -1$, then the complement of the algebraic torus in V_2, consists of 14 toric surfaces 8 and 6 of which are biholomorphic to S_6 and $\mathbf{P}^1 \times \mathbf{P}^1$, respectively (see Figure 5).

(3) If $c_{1,2} = c_{2,1} = c_{1,3} = c_{3,1} = 0$, $c_{2,3} = c_{3,2} = -1$ then $V_3 \simeq \mathbf{P}^1 \times S_6$.

(4) If $c_{1,2} = c_{2,1} = c_{1,3} = c_{3,1} = 0$, $c_{2,3} = -1$, $c_{3,2} = -2$ then $V_4 \simeq \mathbf{P}^1 \times S_8$.

(5) If $c_{1,2} = c_{2,1} = c_{1,3} = c_{3,1} = 0$, $c_{2,3} = -1$, $c_{3,2} = -3$ then $V_5 \simeq \mathbf{P}^1 \times S_{12}$.

(6) If $c_{i,j} = 0$ for all i, j, then $V_6 \simeq \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.

(7) If $(p_\lambda)_{\mathbf{R}}(\sigma)$ is simplical, $v(\mu_i) \cdot v(\mu_j) = 0$ for $1 \leq i < j \leq 3$ and $u_1 = \mathbf{f}_1$, $u_2 = \mathbf{f}_1 + 2\mathbf{f}_2$, $u_3 = \mathbf{f}_3$ for a basis $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_r\}$ of N, then $V_7 \simeq \mathbf{P}^1 \times \bigl(\mathbf{P}^1 \times \mathbf{P}^1/(-1,-1)\bigr)$.

5. Examples with quadratic C

We fix $r = 4$, throughout the rest of this paper.

Example 1. Let $P(x_1, x_2, x_3, x_4) = -x_1^2 - x_2^2 - x_3^2 + 7x_4^2$. Let σ be the cone generated by the following six elements in N.

$$u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ u_2 = \begin{bmatrix} 7 \\ 7 \\ 0 \\ 4 \end{bmatrix}, \ u_3 = \begin{bmatrix} 7 \\ 7 \\ 7 \\ 5 \end{bmatrix}, \ u_4 = \begin{bmatrix} 14 \\ 7 \\ 0 \\ 6 \end{bmatrix}, \ u_5 = \begin{bmatrix} 21 \\ 7 \\ 7 \\ 9 \end{bmatrix}, \ u_6 = \begin{bmatrix} 7 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

Let C be the connected component of C_P containing u_1. Then $\sigma \setminus \{0\} \subset C$. Let

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \ v_4 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ v_5 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

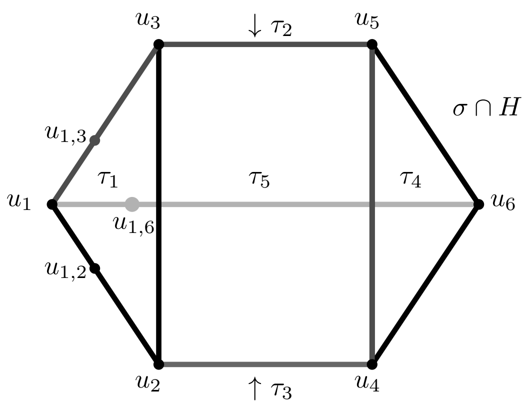


Figure 6

Then $\tau_i := \sigma \cap H_{v_i}$ $(i = 1, \ldots, 5)$ are 3-dimensional faces of σ (see Figure 6 which shows the intersection with a hyperplane H). Moreover, we see by Proposition 5 and easy calculation that $v(\tau_i) = v_i$ satisfy the conditions 2, 3 of Theorem 8. Let Σ be the fan in Theorem 8 defined for this σ. Then $V(\lambda)$ are singularities in $T_N\mathrm{emb}(\Sigma)$ for all cones λ in Σ with $\dim \lambda \geq 2$. Noting that $\sigma^\vee$ is spanned by $i(v_1)$, $i(v_2)$, ..., $i(v_5)$, where $i : N \to M$ is the homomorphism satisfying $\langle i(x), y\rangle = B_P(x, y)$, we see that all 3-dimensional faces of $\sigma^\vee$ are non-singular. Let $\lambda = \mathbf{R}_{\geq 0}u_1$ and let

$$u_{1,2} = \begin{bmatrix}1\\1\\0\\1\end{bmatrix},\ u_{1,3} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix},\ u_{1,6} = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}.$$

Then $\{u_1, u_{1,2}, u_{1,3}, u_{1,6}\}$ is a basis of N and $(p_\lambda)_\mathbf{R}(\sigma) = \mathbf{R}_{\geq 0}p_\lambda(u_{1,2}) + \mathbf{R}_{\geq 0}p_\lambda(u_{1,3}) + \mathbf{R}_{\geq 0}p_\lambda(u_{1,6})$. Moreover, we see by easy calculation that the relations $u_{1,2} + \gamma_{v_2}u_{1,2} - u_{1,3} - u_{1,6} = 0$, $u_{1,3} + \gamma_{v_3}u_{1,3} - 2u_{1,2} = 0$ and $u_{1,6} + \gamma_{v_1}u_{1,6} - u_1 - u_{1,2} = 0$ hold. Hence $V(\lambda)$ is biholomorphic to V_{1a} in the previous section. Since $v_1 \cdot v_3 = v_1 \cdot v_5 = 0$, $v_3 \cdot v_5 = 1$, $v_3 \cdot v_3 = -1$ and $v_5 \cdot v_5 = -2$, $V(\mathbf{R}_{\geq 0}u_2)$ is biholomorphic to V_4. We see by similar caculation that $V(\mathbf{R}_{\geq 0}u_i)$ are biholomorphic to V_2, V_{1a}, V_2 and V_4 for $i = 3, 4, 5$ and 6, respectively.

Example 2. Let $P(x_1, x_2, x_3, x_4) = -x_1^2 - x_2^2 - x_3^2 + 15x_4^2$. Then the cone σ defined by v_1, v_2, ..., v_6, satisfies the conditions of Theorem 8, where

$$v_1 = \begin{bmatrix}-1\\1\\0\\0\end{bmatrix},\ v_2 = \begin{bmatrix}0\\-1\\1\\0\end{bmatrix},\ v_3 = \begin{bmatrix}0\\0\\-1\\0\end{bmatrix},\ v_4 = \begin{bmatrix}5\\0\\0\\1\end{bmatrix},\ v_5 = \begin{bmatrix}3\\3\\0\\1\end{bmatrix},\ v_6 = \begin{bmatrix}3\\2\\2\\1\end{bmatrix}$$

(see Figure 7). We can verify that the divisors corresponding to the vertices attached ⓘ are biholomorphic to V_i in the previous section. For example, $v_2 \cdot v_4 = v_2 \cdot v_6 = v_4 \cdot v_6 = 0$, $(\mathbf{R}w_2 + \mathbf{R}w_i) \cap M = \mathbf{Z}w_2 + \mathbf{Z}w_i$ for $i = 4, 6$ and $[(\mathbf{R}w_4 + \mathbf{R}w_6) \cap M : \mathbf{Z}w_4 + \mathbf{Z}w_6] = 2$, where w_i $(i = 2, 4, 6)$ are the elements in M satisfying $\langle w_2, x\rangle = B_P(v_2, x)$, $\langle w_4, x\rangle = \frac{1}{5}B_P(v_4, x)$ and $\langle w_6, x\rangle = B_P(v_6, x)$. Hence $V(\tau_2 \cap \tau_4 \cap \tau_6)$ is biholomorphic to V_7, where $\tau_i = \sigma \cap H_{v_i}$.

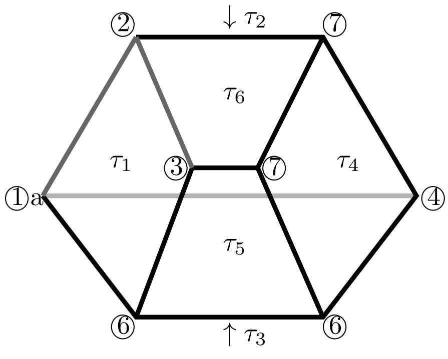


Figure 7

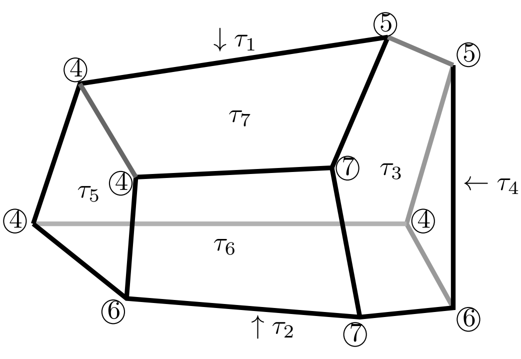


Figure 8

Example 3. Let $P(x_1, x_2, x_3, x_4) = -3x_1^2 - 3x_2^2 - 5x_3^2 + x_4^2$. Then the cone σ defined by v_1, v_2, ..., v_6, where

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix},$$

$$v_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 0 \\ 5 \\ 6 \\ 15 \end{bmatrix}, \quad v_7 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

(see Figure 8).

6. An example with non-quadratic *C*

We fix a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ of N. Let γ_i be the elements in $\mathrm{GL}(N)$ defined by the following relations for $i = 1, 2, 3, 4$. $\gamma_i \mathbf{e}_j = \mathbf{e}_j$ if $i \neq j$ and

$$\gamma_1 \mathbf{e}_1 = -\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3,\ \gamma_2 \mathbf{e}_2 = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_4,\ \gamma_3 \mathbf{e}_3 = \mathbf{e}_1 - \mathbf{e}_3 + \mathbf{e}_4,\ \gamma_4 \mathbf{e}_4 = 2\mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4.$$

Then $\Gamma_6 = \langle \gamma_i \mid i = 1, 2, 3, 4 \rangle$ is a Coxeter group with the relations: $\gamma_i^2 = 1$ and

$$(*)\ \ (\gamma_1\gamma_2)^3 = (\gamma_3\gamma_4)^3 = (\gamma_1\gamma_3)^4 = (\gamma_2\gamma_4)^4 = (\gamma_1\gamma_4)^2 = (\gamma_2\gamma_3)^2 = 1.$$

Hence the Dynkin diagram of Γ_6 is Figure 9 (see [2, 2.3] for the definition of Dynkin diagram). Let $\sigma = \mathbf{R}_{\geq 0}\mathbf{e}_1 + \mathbf{R}_{\geq 0}\mathbf{e}_2 + \mathbf{R}_{\geq 0}\mathbf{e}_3 + \mathbf{R}_{\geq 0}\mathbf{e}_4$ and let τ_i be the 3-dimensional face of σ which does not contain $\mathbf{e}_i$ for each i. Then γ_i is a reflection with respect to the hyperplane containing τ_i. Moreover, the entries a_{ij} of the Cartan matrix in [10], are equal to $-c_{ji}$ if $i \neq j$, where c_{ji} are the coefficients in the above relations $\gamma_j \mathbf{e}_j = \sum c_{ji} \mathbf{e}_i$, because $2\mathbf{e}_j - \sum_{i \neq j} c_{ji} \mathbf{e}_i$ is an eigenvector of γ_j with the eigenvalue -1. Hence $a_{14} = a_{41} = a_{23} = a_{32} = 0$, $a_{12} \cdot a_{21} = a_{34} \cdot a_{43} = 1$, $a_{13} \cdot a_{31} = a_{24} \cdot a_{42} = 2$. Therefore, $C_6 = \bigcup_{\gamma \in \Gamma_6} \gamma\sigma \setminus \{0\}$ is an open strongly convex cone in $N_{\mathbf{R}}$ and $\Sigma_6 = \{\gamma\tau \mid \gamma \in \Gamma_6, \tau \prec \sigma\}$ is a Γ_6-invariant fan with $|\Sigma_6| = C_6 \cup \{0\}$, by [10, Theorem 1]. Moreover, C_6 is not quadratic, by [10, Theorem 6]. Since σ is non-singular, so is $T_N\mathrm{emb}(\Sigma_6)$. The 3-dimensional toric variety $V(\mathbf{R}_{\geq 0}\mathbf{e}_i)$ is biholomorphic to V_{1a} (resp. V_{1b}) in Section 4 for $i = 2, 3$ (resp. $1, 4$). The intersection $V(\mathbf{R}_{\geq 0}\mathbf{e}_i) \cap V(\mathbf{R}_{\geq 0}\mathbf{e}_j) = V(\mathbf{R}_{\geq 0}\mathbf{e}_i + \mathbf{R}_{\geq 0}\mathbf{e}_j)$ is the toric surface corresponding to the Coxeter group generated by $\{\gamma_k, \gamma_l\}$ for $\{k, l\} = \{1, 2, 3, 4\} \setminus \{i, j\}$. Hence it is biholomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ if $(i, j) = (2, 3), (1, 4)$, S_6 if $(i, j) = (3, 4), (1, 2)$ and S_8 if $(i, j) = (2, 4), (1, 3)$ by $(*)$. Note that $V(\mathbf{R}_{\geq 0}\mathbf{e}_i) \cap V(\mathbf{R}_{\geq 0}\mathbf{e}_j)$ is biholomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, if and only if $V(\mathbf{R}_{\geq 0}\mathbf{e}_i)$ and $V(\mathbf{R}_{\geq 0}\mathbf{e}_j)$ are biholomorphic.

Remark. Let Γ_6', Σ_6' and C_6' be the subgroup of $\mathrm{GL}(N)$, the fan and the cone in $N_{\mathbf{R}}$, respectively, obtained by transposing the coefficients $c_{2,4} = 1$ and $c_{4,2} = 2$ in the above relations $\gamma_i \mathbf{e}_i = \sum c_{ij} \mathbf{e}_j$. Then the irreducible components of $T_N\mathrm{emb}(\Sigma_6') \setminus T_N$ are isomorphic to those of $T_N\mathrm{emb}(\Sigma_6) \setminus T_N$. However, they intersect to each other in a different way. $V(\mathbf{R}_{\geq 0}\mathbf{e}_i)$ are biholomorphic to V_{1a} (resp. V_{1b}) for $i = 1, 2$ (resp.

$3, 4)$. Hence $V(\mathbf{R}_{\geq 0}\mathbf{e}_i) \cap V(\mathbf{R}_{\geq 0}\mathbf{e}_j)$ is biholomorphic to S_6, if and only if $V(\mathbf{R}_{\geq 0}\mathbf{e}_i)$ and $V(\mathbf{R}_{\geq 0}\mathbf{e}_j)$ are biholomorphic. However, the following consideration for (C_6, Γ_6) holds also for (C'_6, Γ'_6), because the relations in $(*)$ do not change.

Hereafter, we simply write Γ, Σ and C for Γ_6, Σ_6 and C_6, respectively.

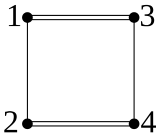


Figure 9

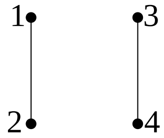


Figure 10

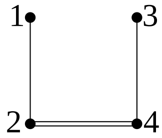


Figure 11

THEOREM 9. *There exists a subgroup* Γ^0 *of* Γ *of index* 48 *which acts on* D_C *freely. Conversely, if a subgroup* Γ' *of* Γ *acts on* D_C *freely, then* Γ' *is of index at least* 48.

Let $\Gamma^i = \langle \gamma_j \mid 1 \leq j \leq 4, j \neq i \rangle$ for each i. Then Γ^i is the stabilizer of $\mathbf{R}_{\geq 0}\mathbf{e}_i$ and $|\Gamma^i| = 48$. Hence the second assertion in the above theorem holds. Let $\Delta = \{p_C(\tau \setminus \{0\}) \mid \tau \in \Sigma, \tau \neq \{0\}\}$. Then Δ is a Γ-invariant tetrahedral decomposition of D_C. If we get Γ^0 in the above theorem, then Δ/Γ^0 is a tetrahedral decomposition of the 3-dimensional compact topological manifold D_C/Γ^0 consisting of 48 tetrahedra. Since Δ/Γ^0 has $48 \cdot 4/|\Gamma^i| = 4$ vertices, there exists a resolution of the cusp singularity $\mathrm{Cusp}(C, \Gamma^0)$ with an exceptional set consisting of 4 irreducible components. The rest of this section is devoted to the proof of the first assertion in the above theorem.

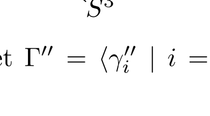
Let γ'_i be the elements in $\mathrm{GL}(N)$ defined by the following relations for $i = 1, 2, 3, 4$. $\gamma'_i \mathbf{e}_j = \mathbf{e}_j$ if $i \neq j$ and

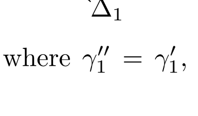
$$\gamma'_1\mathbf{e}_1 = -\mathbf{e}_1 + \mathbf{e}_2,\ \gamma'_2\mathbf{e}_2 = \mathbf{e}_1 - \mathbf{e}_2,\ \gamma'_3\mathbf{e}_3 = -\mathbf{e}_3 + \mathbf{e}_4,\ \gamma'_4\mathbf{e}_4 = \mathbf{e}_3 - \mathbf{e}_4.$$

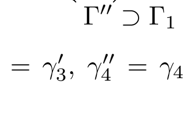
Then $\Gamma' = \langle \gamma'_i \mid i = 1, 2, 3, 4 \rangle$ is a Coxeter group with the relations: $\gamma'^2_i = 1$ and

$$(\gamma'_1\gamma'_2)^3 = (\gamma'_3\gamma'_4)^3 = (\gamma'_1\gamma'_3)^2 = (\gamma'_2\gamma'_4)^2 = (\gamma'_1\gamma'_4)^2 = (\gamma'_2\gamma'_3)^2 = 1$$

Hence the Dynkin diagram of Γ' is Figure 10, $\Gamma' \simeq D_3 \times D_3$ and there exists a surjective homomorphism $q : \Gamma \to \Gamma'$ sending γ_i to γ'_i. Let $\Delta' = \{p(\gamma'\tau \setminus \{0\}) \mid \gamma' \in \Gamma', \tau \prec \sigma, \tau \neq \{0\}\}$, where $p : N_{\mathbf{R}} \setminus \{0\} \to S^3$ is the natural projection. Then Δ' is a tetrahedral decomposition of S^3 with 36 tetrahedra. Let $\tilde{f} : C \cup \{0\} \to N_{\mathbf{R}}$ be the piecewise linear map defined by $\tilde{f}(x) = q(\gamma)\gamma^{-1}x$, if x is in $\gamma\sigma$ for an element γ in Γ. Then $\tilde{f}$ induces a Galois covering $f : D_C \to S^3$ with $f(\gamma x) = q(\gamma)f(x)$ for any element γ in Γ, ramifying only along $\Xi_{13} \cup \Xi_{24}$, where $\Xi_{ij} = \bigcup_{\gamma' \in \Gamma'} p(\gamma'(\mathbf{R}_{\geq 0}\mathbf{e}_i + \mathbf{R}_{\geq 0}\mathbf{e}_j) \setminus \{0\})$, because $\langle \gamma_i, \gamma_j \rangle$ are the stabilizers of $\mathbf{R}_{\geq 0}\mathbf{e}_k + \mathbf{R}_{\geq 0}\mathbf{e}_l$, where $\{k, l\} = \{1, 2, 3, 4\} \setminus \{i, j\}$, $q((\gamma_2\gamma_4)^2) = q((\gamma_1\gamma_3)^2) = 1$ and the restriction of q to $\langle \gamma_i, \gamma_j \rangle$ is an isomorphism if $(i, j) \neq (1, 3), (2, 4)$. Moreover, Δ is the pull-back of Δ' under f.







Let $\Gamma'' = \langle \gamma''_i \mid i = 1, 2, 3, 4 \rangle$, where $\gamma''_1 = \gamma'_1$, $\gamma''_2 = \gamma_2$, $\gamma''_3 = \gamma'_3$, $\gamma''_4 = \gamma_4$. Then

Γ'' is a Coxeter group whose Dynkin diagram is Figure 11 and there exist surjective homomorphisms $q_1 : \Gamma \to \Gamma''$ sending γ_i to γ''_i and $q'_1 : \Gamma'' \to \Gamma'$ sending γ''_i to γ'_i with $q = q'_1 \circ q_1$. We can define Galois coverings $f_1 : D_C \to S^3$ and $f'_1 : S^3 \to S^3$ such that $f'_1(\gamma'' x) = q'_1(\gamma'')f'_1(x)$ for any element γ'' in Γ'' and that $f'_1 \circ f_1 = f$, in a similar way as f. Then f'_1 ramifies only along Ξ_{13}, $\mathrm{Gal}(f'_1) = \ker(q'_1)$ and $\Delta_1 = \{p(\gamma''\tau \setminus \{0\}) \mid \gamma'' \in \Gamma'', \tau \prec \sigma, \tau \neq \{0\}\}$ is the pull-back of Δ' under f'_1. Let $\gamma''_0 = \gamma''_1\gamma''_2\gamma''_3\gamma''_4$.

Lemma. There exists a normal subgroup Γ_1 of $\ker(q'_1)$ acting on S^3 freely with $\ker(q'_1)/\Gamma_1 \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$, $\gamma''^3_0 \in \Gamma_1$ and $\gamma''_0\Gamma_1\gamma''^{-1}_0 = \Gamma_1$.

Proof. Let $\mathbb{P}$ be the convex hull of the 24 points

$$\begin{pmatrix}\pm2\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\\pm2\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\\pm2\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\\pm2\end{pmatrix}, \quad \begin{pmatrix}\pm1\\\pm1\\\pm1\\\pm1\end{pmatrix}$$

in $\mathbf{R}^4$. Then the boudary $\partial\mathbb{P}$ of $\mathbb{P}$ consists of 24 octahedra which are on the hyperplanes defined by $\pm x_i \pm x_j = 2$ $(1 \leq i < j \leq 4)$, and is a regular polyhedron of type $(3,4,3)$ (see [1, 8.2]). For example, an octahedron has 6 vertices ${}^t(2,0,0,0)$, ${}^t(0,2,0,0)$, ${}^t(1,1,\pm1,\pm1)$. Let $\square$ be the barycentric subdivision of the octahedral decomposition $p(\partial\mathbb{P})$ of S^3 which is the image of $\partial\mathbb{P}$ under the projection $p : \mathbf{R}^4 \setminus \{0\} \to S^3$. Let $h : S^3 \to S^3$ be the homeomorphism induced by the linear transformation $\tilde{h}$ sending $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$ to ${}^t(1,1,0,0)$, ${}^t(2,1,1,0)$, ${}^t(1,1,1,1)$ and ${}^t(2,2,2,0)$, respectively. Then $h\Delta_1$ coincides with $\square$, because $\tilde{h}(\gamma''_1\mathbf{e}_1) = {}^t(1,0,1,0)$, $\tilde{h}(\gamma''_2\mathbf{e}_2) = {}^t(1,2,1,0)$, $\tilde{h}(\gamma''_3\mathbf{e}_3) = {}^t(1,1,1,-1)$ and $\tilde{h}(\gamma''_4\mathbf{e}_4) = {}^t(3,1,1,1)$ (see Figure 12).

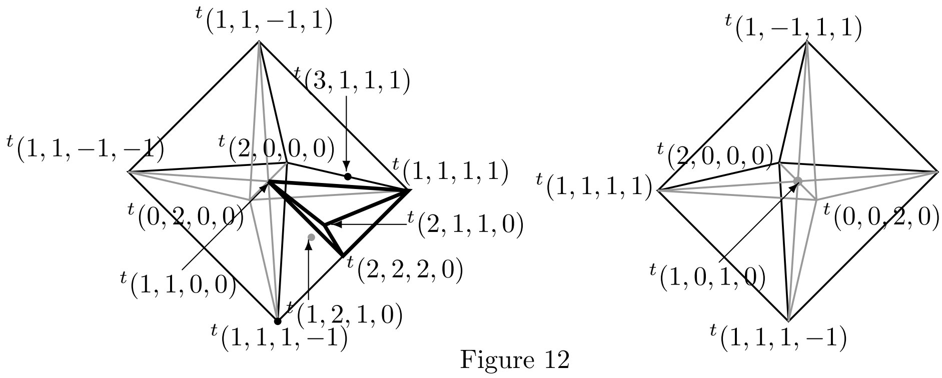


Figure 12

Moreover, $h(f'^{-1}_1(\Xi_{13}))$ is the union of the diagonals of the octahedra on $p(\partial\mathbb{P})$. Since the barycentric subdivision of an octahedron has 48 tetrahedra, $|\Gamma''| = 24 \cdot 48 = 1152$. Since $\ker(q'_1)$ is generated by the conjugates of $(\gamma''_2\gamma''_4)^2$, whose fixed points are contained in $f'^{-1}_1(\Xi_{13})$ and $|\ker(q'_1)| = |\Gamma''|/|\Gamma'| = 1152/36 = 32$, $\tilde{h}\ker(q'_1)\tilde{h}^{-1}$ consists of the following 32 matrices, where $\epsilon_i = \pm1$ and $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1$.

$$\begin{pmatrix}\epsilon_1&0&0&0\\0&\epsilon_2&0&0\\0&0&\epsilon_3&0\\0&0&0&\epsilon_4\end{pmatrix}, \quad \begin{pmatrix}0&\epsilon_1&0&0\\\epsilon_2&0&0&0\\0&0&0&\epsilon_3\\0&0&\epsilon_4&0\end{pmatrix}, \quad \begin{pmatrix}0&0&\epsilon_1&0\\0&0&0&\epsilon_2\\\epsilon_3&0&0&0\\0&\epsilon_4&0&0\end{pmatrix}, \quad \begin{pmatrix}0&0&0&\epsilon_1\\0&0&\epsilon_2&0\\0&\epsilon_3&0&0\\\epsilon_4&0&0&0\end{pmatrix}.$$

Note that the fixed points of all matrices of order 2 in the above except $-I_4$, are contained in the diagonals of the octahedra and that any one of order 4 in the above is the product of two of order 2. The set consisting of $\pm I_4$, $\pm A$, $\pm B$ and $\pm C$ is a normal subgroup of $\tilde{h}\ker(q_1')\tilde{h}^{-1}$ acting on S^3 freely, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $J = \tilde{h}\gamma_0''\tilde{h}^{-1}$. Then

$$J = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}.$$

Hence $J^3 = -B$, $JAJ^{-1} = -A$, $JBJ^{-1} = B$ and $JCJ^{-1} = -C$. Since $|\ker(q_1')/\Gamma_1| = 4$ and $X^2 = -I_4$ for any element X of order 4 in $\tilde{h}\ker(q_1')\tilde{h}^{-1}$, $\ker(q_1')/\Gamma_1 \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$. $\square$

Let $T_1 = S^3/\Gamma_1$ and let $g_1' : T_1 \to S^3$ be the Galois covering induced by f_1'. Then g_1' ramifies only along Ξ_{13}. Let $h_1 : D_C \to T_1$ be the composite of f_1 and the quotient map $S^3 \to T_1$ under Γ_1. Then h_1 ramifies only along $g_1'^{-1}(\Xi_{24})$ and $f = g_1' \circ h_1$. Moreover, γ_0'' induces an automorphism δ_1 on T_1 with $|\delta_1| = 3$, by the above lemma. Let $\gamma_0' = \gamma_1'\gamma_2'\gamma_3'\gamma_4'$. Then γ_0' has no fixed points on S^3 and $q_1'(\gamma_0'') = \gamma_0'$. Hence $g_1' \circ \delta_1 = \gamma_0' \circ g_1'$. In a similar way, we obtain Galois coverings $g_2' : T_2 \to S^3$ ramifying only along Ξ_{24}, $h_2 : D_C \to T_2$ ramifying only along $g_2'^{-1}(\Xi_{13})$ with $f = g_2' \circ h_2$ and an automorphism δ_2 on T_2 with $|\delta_2| = 3$ such that $g_2' \circ \delta_2 = \gamma_0' \circ g_2'$.

$$D_C \to T = T_1 \times_{S^3} T_2 \to T_0 = T/G_0 \to T_0/\langle\delta_0\rangle = D_C/\Gamma^0 \to S^3$$

Now, to show the existence of a subgroup Γ^0 in the theoerm, we construct covering maps as above, where the left three arrows do not ramify and the right one ramifies along $\Xi_{13} \cup \Xi_{24}$. Let $T = T_1 \times_{S^3} T_2$ be the fiber product of g_1' and g_2'. Then T is a topological manifold, because $\Xi_{13} \cap \Xi_{24} = \varnothing$. Since $\mathrm{Gal}(g_i') \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$, any bijection between $\mathrm{Gal}(g_1') \setminus \{1\}$ and $\mathrm{Gal}(g_2') \setminus \{1\}$ induces an isomorphism. Hence there exists an isomorphism $\phi : \mathrm{Gal}(g_1') \simeq \mathrm{Gal}(g_2')$ such that $\phi(\delta_1\gamma\delta_1^{-1}) = \delta_2\phi(\gamma)\delta_2^{-1}$ for any element γ in $\mathrm{Gal}(g_1')$. Let $G_0 = \{(\gamma, \phi(\gamma)) \mid \gamma \in \mathrm{Gal}(g_1')\}$. Then G_0 has no fixed points on T, because $\Xi_{13} \cap \Xi_{24} = \varnothing$. Let $T_0 = T/G_0$ and let $g_0' : T_0 \to S^3$ be the covering induced by the natural projection $T \to S^3$. Then $\deg g_0' = 4$, because $\deg g_i' = 4$. Hence the pull-back of Δ' under g_0', consists of $36 \cdot 4 = 144$ tetrahedra. Let $h : D_C \to T_0$ be the composite of the map (h_1, h_2) and the quotient map $T \to T_0$. Then h is a surjective unramified covering, because it does not ramify along $g_0'^{-1}(\Xi_{13} \cup \Xi_{24})$ and T_0 is a topological manifold. Since $(\delta_1, \delta_2)G_0(\delta_1, \delta_2)^{-1} = G_0$, (δ_1, δ_2) induces an automorphism δ_0 on T_0 with $g_0' \circ \delta_0 = \gamma_0' \circ g_0'$. Since γ_0' has no fixed points on S^3, so does δ_0 on T_0. Hence the composite of h and the quotient map $T_0 \to T_0/\langle\delta_0\rangle$, is the quotient map under a subgroup of Γ with the index $144/3 = 48$ acting on D_C freely.

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