Fans consisting of infinitely many non-singular cones

Hiroyasu Tsuchihashi

Abstract

We give a method of constructing fans as the title and groups acting properly discontinuously on certain open sets of the associated toric varieties. As the quotient spaces, we obtain examples of cusp singularities of dimension greater than 2, non-isolated singularities, degenerating families of compact complex manifolds of Kodaira dimension 0 and compact complex manifolds of any dimension greater than 3 with infinite cyclic fundamental groups.

2000 Mathematics Subject Classification. Primary 14J17; Secondary 32S05

key words: fan, toric variety, cusp singularity

0 Introduction

Let r be an integer greater than 2 and let $N = \mathbf{Z}^r$. Degenerating families of Abelian varieties of a restricted type and cusp singularities are constructed from fans Σ in N and subgroups Γ of $GL(N)$ satisfying the following condition (see for instance [3], [4] and [5]).

(F) Σ is Γ-invariant, i.e., $\gamma\sigma$ is in Σ for any σ in Σ and any γ in Γ, and Σ/Γ is finite.

In [5], we constructed 3-dimensional fans and groups as above using triangulations of compact topological surfaces to both sides of whose edges integers are attached. In this paper, using simplicial decompositions of topological spaces which may not be topological manifolds, we give a method of constructing fans in N and subgroups of $GL(N)$ which satisfy the above condition (F) and give examples of singularities, degenerating families of compact complex manifolds of Kodaira dimension 0 and compact complex manifolds with infinite cyclic fundamental groups.

Let Δ be a topological space obtained by gluing finitely many $(r-1)$-dimensional simplices together (we give an exact definition in Section 1). Here Δ may have boundaries and its interior may not be a topological manifold. Assume that integers are attached to all vertices of $(r-2)$-dimensional simplices which are not on the boundaries of Δ. In Section 2, we construct a Galois covering $f : \widetilde{\Delta} \to \Delta$ whose restriction to $f^{-1}(T)$ is a universal covering, where T is the complement of the union of $(r-2)$-dimensional simplices on the boundary of Δ, some $(r-3)$-dimensional simplices and all low-dimensional simplices, and define a map h from the set $\widetilde{\Delta}^0$ of vertices of $\widetilde{\Delta}$ to $N \setminus \{0\}$ as follows. Choose an $(r-1)$-dimensional simplex $\alpha = \overline{v_1 v_2 \cdots v_r}$ of $\widetilde{\Delta}$ and a basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r\}$ of N, and let $h(v_i) = \mathbf{e}_i$ for $1 \le i \le r$. Let $\alpha' = \overline{w_1 v_2 \cdots v_r}$ be the $(r-1)$-dimensional symplex adjacent to α at the $(r-2)$-dimensional face $\beta = \overline{v_2 \cdots v_r}$, and let w_1 be the vertex of α' which is not on β. Then we define $h(w_1)$ by the equality

$$h(v_1) + h(w_1) + \sum_{i=2}^{r} a_i h(v_i) = 0,$$

where a_i are the integers attached to the vertices $f(v_i)$ of the $(r-2)$-dimensional simplex $f(\beta)$ of Δ. If the integers attached to Δ satisfy certain conditions ((M), (C) in Section 1), then repeating the above process, we have a map $h : \widetilde{\Delta}^0 \to N \setminus \{0\}$. Then

$$\Sigma(\Delta) := \{\mathbf{R}_{\ge 0} h(v_1) + \cdots + \mathbf{R}_{\ge 0} h(v_l) \mid \overline{v_1 \cdots v_l} \text{ are simplices of } \widetilde{\Delta}\} \bigcup \{\{0\}\}$$

is a set of non-singular cones. In Theorem 12, we give a sufficient condition that $\Sigma(\Delta)$ becomes a fan.

Assume that $\Sigma(\Delta)$ is a fan and let Y be the associated toric variety. Then the irreducible components of the complement of the algebraic torus in Y and their intersections are also toric varieties. In Section 3, we study the structure of them. There exists a homomorphism $\rho : \mathrm{Gal}(\widetilde{\Delta}/\Delta) \to GL(N)$ with $h(\gamma v) = \rho(\gamma)h(v)$ for all vertices v of $\widetilde{\Delta}$ and all elements γ in $\mathrm{Gal}(\widetilde{\Delta}/\Delta)$, and its image $\Gamma := \mathrm{Im}(\rho)$ acts on Y. In Section 4, we show that there exists an open set $\widetilde{U}$ of Y on which Γ acts properly discontinuously. Let $U = \widetilde{U}/\Gamma$. In Section 5, we give a sufficient condition that U has a fibration. As an application, we can obtain a degenerating family of elliptic curves with a base space of any dimension. In Section 6, we consider the case that an anyalytic subset of U contracts to singularities. Theorem 23 gives a sufficient condition that a compact analytic subset of U contracts to a point. As an application, we can obtain a singularity with a resolution whose exceptional set consists of rational curves intersecting as the edges of a polyhedron. Theorem 24 gives a sufficient condition that there exists a holomorphic map from an open set of U to an analytic space which may have non-isolated singularities. As an application, we obtain a non-isolated singularity with a resolution whose exceptional set consists of non-compact elliptic surfaces intersecting along singular fibers of type I_2. In Section 7, we show that for any negative integer e there exist 3-dimensional cusp singularities with resolutions whose exceptional sets are irreducible and their dual graphs are triangulations of compact topological surfaces with the Euler number e. In Section 8, we construct an example of Δ of any dimension greater than 3 which gives a compact complex manifold with an infinite cyclic fundamental group. In Section 9, we consider deformations of actions of Γ by torus actions. If a map $t : \Gamma \to T_N$ satisfies a certain condition, then $\Gamma(t) := \{t(\gamma) \circ \gamma \mid \gamma \in \Gamma\}$ is a group isomorphic to Γ and acts on Y. We study the structure of the quotient space of an open set of Y under $\Gamma(t)$, in some cases.

1 Simplicial systems and Z-weights

Let N be as in Introduction and let $M = \mathrm{Hom}(N, \mathbf{Z})$ with a canonical pairing $\langle\ ,\ \rangle : M \times N \to \mathbf{Z}$. Let A^{r-1} be a non-empty set of $(r-1)$-dimensional simplices and let $A = \bigcup_{i=0}^{r-1} A^i$, where A^i is the set of i-dimensional faces of simplices in A^{r-1}. Let B be a non-empty subset of $A^{r-2} \times A^{r-2}$ satisfying the following conditions (i)-(iv).

(i) B does not contain (β, β) for any β in A^{r-2}.

(ii) If $(\beta_1, \beta_2) \in B$, then $(\beta_2, \beta_1) \in B$.

(iii) $|\{\beta' \in A^{r-2} \mid (\beta, \beta') \in B\}| \leq 1$ for any β in A^{r-2}.

(iv) For any two elements α and α' in A^{r-1}, there exist elements $\beta'_1, \beta_2, \beta'_2, \ldots, \beta'_{k-1}, \beta_k$ in A^{r-2} such that $(\beta'_i, \beta_{i+1}) \in B$ for $1 \leq i < k$, $G(\beta'_1) = \alpha$, $G(\beta_i) = G(\beta'_i)$ for $2 \leq i < k$, $G(\beta_k) = \alpha'$, where $G(\beta)$ is the element in A^{r-1} of which β is a face.

We assume that for each (β_1, β_2) in B there exists a bijection $l_{(\beta_1,\beta_2)} : \beta_1 \to \beta_2$ expressed as $l_{(\beta_1,\beta_2)}(t_1v_1 + \cdots + t_{r-1}v_{r-1}) = t_1w_{k_1} + \cdots + t_{r-1}w_{k_{r-1}}$, where $\beta_1 = \overline{v_1 \cdots v_{r-1}}$, $\beta_2 = \overline{w_1 \cdots w_{r-1}}$ and $(k_1, k_2, \ldots, k_{r-1})$ is a permutation of $(1, 2, \ldots, r-1)$. Moreover, we assume that $\{l_b\}_{b \in B}$ satisfies the following:

(v) $l_{(\beta_2,\beta_1)} = l^{-1}_{(\beta_1,\beta_2)}$ for any (β_1, β_2) in B.

Let Δ be the topological space obtained by gluing β_1 and β_2 together using $l_{(\beta_1,\beta_2)}$ for all (β_1, β_2) in B, i.e., $\Delta = \sqcup_{\alpha \in A^{r-1}} \alpha / \sim$, where $p \sim q$ if there exist elements β'_1, β_2, β'_2, $\ldots$, β_k in A^{r-2} such that $(\beta'_i, \beta_{i+1}) \in B$ for $1 \leq i < k$, that $(l_{(\beta'_i,\beta_{i+1})} \circ \cdots \circ l_{(\beta'_1,\beta_2)})(p) \in \beta'_{i+1}$ for $1 \leq i < k-1$ and that $(l_{(\beta'_{k-1},\beta_k)} \circ \cdots \circ l_{(\beta'_1,\beta_2)})(p) = q$. Then Δ is connected, by the condition (iv). Let $I : \sqcup_{\alpha \in A} \alpha \to \Delta$ be the quotient map.

Definition. We call a triple $(A, B, \{l_b\}_{b \in B})$ satisfying (i)-(v), a simplicial system and the above Δ, the topological space induced from $(A, B, \{l_b\}_{b \in B})$. We call the image $I(\beta)$ of an element β in A^i under I, an i-dimensional slice.

Let Δ^i be the set of i-dimensional slices. Let $A_{\text{in}}^{r-2} = \text{pr}_1(B)$ be the image of B under the projection $\text{pr}_1 : A^{r-2} \times A^{r-2} \to A^{r-2}$ and let $\Delta_{\text{in}}^{r-2} = \{I(\beta) \mid \beta \in A_{\text{in}}^{r-2}\}$. Then $I(\bigcup_{\alpha \in A^{r-1} \cup A_{\text{in}}^{r-2}} \alpha^o)$ is a topological manifold by the condition (iii), where α^o is the interior of α. Let $A_{\text{bd}}^{r-2} = A^{r-2} \setminus A_{\text{in}}^{r-2}$ and let $\Delta_{\text{bd}}^{r-2} = \{I(\beta) \mid \beta \in A_{\text{bd}}^{r-2}\}$. For a slice $\bar{\beta}$ in Δ and an integer i with $\dim \bar{\beta} < i \leq r-1$, let $n(\bar{\beta}, i) = |\{(\beta, \alpha) \in A^{\dim \bar{\beta}} \times A^i \mid \beta \prec \alpha, I(\beta) = \bar{\beta}\}|$ and let $\Delta^i(\bar{\beta}) = \{I(\alpha) \mid \beta \prec \alpha \in A^i, I(\beta) = \bar{\beta}\}$, where the notation $\beta \prec \alpha$ implies that β is a face of α. Let $\Delta_{\text{in}}^{r-3} = \{\bar{\beta} \in \Delta^{r-3} \mid \Delta^{r-2}(\bar{\beta}) \subset \Delta_{\text{in}}^{r-2},\ n(\bar{\beta}, r-1) < \infty\}$ and let $\Delta_{\text{bd}}^{r-3} = \Delta^{r-3} \setminus \Delta_{\text{in}}^{r-3}$. Let $A^i(\beta) = \{\alpha \in A^i \mid \beta \prec \alpha\}$ for $\beta \in A$, $\dim \beta < i \leq r-1$. Obviously, $A^{r-1}(\beta) = \{G(\beta)\}$ and the following holds.

Lemma 1. *For each slice $\bar{\tau}$ in Δ_{in}^{r-3}, there exist elements $\tau_1, \ldots, \tau_s$ ($s = n(\bar{\tau}, r-1)$) in A^{r-3}, $\beta_1, \beta_1', \ldots, \beta_s, \beta_s'$ in A^{r-2} satisfying the following: $I^{-1}(\bar{\tau}) = \tau_1 \cup \tau_2 \cup \cdots \cup \tau_s$, $\tau_i \neq \tau_j$ if $i \neq j$, $A^{r-2}(\tau_i) = \{\beta_i, \beta_i'\}$, $(\beta_i', \beta_{i+1}) \in B$ for $1 \leq i \leq s$, where $\beta_{s+1} = \beta_1$, and $l_{(\beta_i', \beta_{i+1})}(\tau_i) = \tau_{i+1}$ for $1 \leq i \leq s$, where $\tau_{s+1} = \tau_1$.*

Definition. A $\mathbf{Z}$-weight of a simplicial system $(A, B, \{l_b\}_{b \in B})$ is a map $\phi : W \to \mathbf{Z}$ from $W = \{(\beta, v) \in A_{\text{in}}^{r-2} \times A^0 \mid v \in \beta\}$ to $\mathbf{Z}$ satisfying the following:
(vi) $\phi(\beta_1, v) = \phi(\beta_2, l_{(\beta_1, \beta_2)}(v))$ for each $(\beta_1, \beta_2) \in B$ and $(\beta_1, v) \in W$.

The following is one of conditions that we can construct a fan from a simplicial system with a $\mathbf{Z}$-weight.

(M) For a slice $\bar{\tau}$ in Δ_{in}^{r-3}, let τ_i, β_i, β_i' be as in Lemma 1, let v_i be the vertex of β_i not contained in τ_i and let $a_i = \phi(\beta_i, v_i)$ for $1 \leq i \leq s$. Then we obtain (M-i) or (M-ii) by repeating the following operation (*) on the sequence $[[a_1, a_2, \ldots, a_s]]$ of the above integers.

(*) Remove an integer a_i equal to -1 and add 1 to the integers $a_{i\pm1}$ on both sides of a_i, where $a_0 = a_s$ and $a_{s+1} = a_1$.

(M-i) $[[1, 1, 1]]$ or $[[a, 0, -a, 0]]$ for an integer a. Moreover, the following conditions (a) and (b) hold.

(a) $l_{(\beta_s', \beta_1)|\tau_s} \circ l_{(\beta_{s-1}', \beta_s)|\tau_{s-1}} \circ \cdots \circ l_{(\beta_1', \beta_2)|\tau_1} = \text{id}$. Hence we may assume that $l_{(\beta_i', \beta_{i+1})}(w_{i,k}) = w_{i+1,k}$ for $1 \leq i \leq s$ and $1 \leq k \leq r-2$, where $\tau_i = \overline{w_{i,1} \ldots w_{i,r-2}}$ and $\beta_{s+1} = \beta_1$, $w_{s+1,k} = w_{1,k}$.

(b) $M_1 M_2 \cdots M_s = I_r$, where

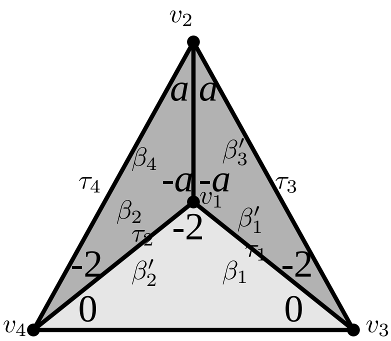
$$M_i = \begin{bmatrix} A_i & O \\ B_i & I_{r-2} \end{bmatrix},\ A_i = \begin{bmatrix} 0 & -1 \\ 1 & -a_i \end{bmatrix},\ B_i = \begin{bmatrix} 0 & -\phi(\beta_i, w_{i,1}) \\ \vdots & \vdots \\ 0 & -\phi(\beta_i, w_{i,r-2}) \end{bmatrix}.$$

(M-ii) $[[a_1', a_2', \ldots, a_l']]$ ($l \geq 1$, $a_j' \leq -2$).

Example 1. $r = 4$, $A^3 = \{\alpha\}$, $B = \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_1), (\alpha_3, \alpha_4), (\alpha_4, \alpha_3)\}$, where $\alpha = \overline{v_1 v_2 v_3 v_4}$ and α_i are the 2-dimensional faces of α which do not contain v_i. Let $l_{(\alpha_4, \alpha_3)}$ be the linear map sending v_1, v_2, v_3 to v_1, v_2, v_4, respectively and let $l_{(\alpha_2, \alpha_1)}$ be the linear map sending v_1, v_3, v_4 to v_2, v_4, v_3, respectively. Then $(A, B, \{l_b\}_{b \in B})$ is a simplicial system. Let $\phi(\alpha_4, v_2) = -\phi(\alpha_4, v_1) = a \in \mathbf{Z}$, $\phi(\alpha_2, v_3) = \phi(\alpha_2, v_4) = 0$, $\phi(\alpha_4, v_3) = \phi(\alpha_2, v_1) = -2$ and we define $\phi(\alpha_1, *)$, $\phi(\alpha_3, *)$ so that (vi) holds (see the left of Figure 1). Then the four edges $\overline{v_1 v_3}$, $\overline{v_1 v_4}$, $\overline{v_2 v_3}$, $\overline{v_2 v_4}$ are mapped under I to one slice of Δ around which ϕ satisfies the condition (M-i) (see the right of Figure 1).

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -a & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & -a & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

While around $I(\overline{v_1 v_2})$ and $I(\overline{v_3 v_4})$, ϕ satisfies the condition (M-ii).



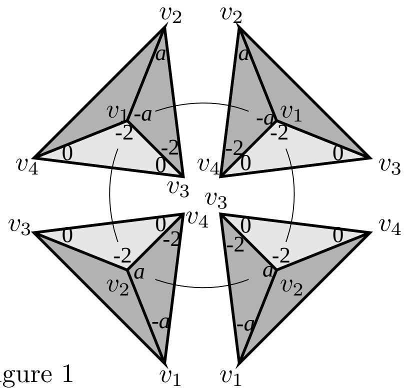


Figure 1

If the condition (M-i) except (b) is satisfied and ϕ is periodic around an $(r-3)$-dimensional slice of Δ, then the following proposition assure that also the condition (b) is satisfied.

Proposition 2. *Assume that the condition (M-i) except (b) is satisfied and that there exists a positive divisor l of s smaller than s such that $M_i = M_j$, if $i \equiv j (\text{mod } l)$. Then $M_1 M_2 \cdots M_s = I_r$.*

Proof. $A_1 A_2 \cdots A_s = I_2$ and $A_1 \cdots A_i \neq I_2$ for $1 \leq i < s$, because

$$\begin{bmatrix} 0 & -1 \\ 1 & -a_{i-1} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -a_{i+1} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -(a_{i-1}+1) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -(a_{i+1}+1) \end{bmatrix}.$$

Hence $A := A_1 \cdots A_l \neq I_2$ and $A^{s/l} = I_2$.

$$M_1 \cdots M_l = \begin{bmatrix} A & O \\ B & I_{r-2} \end{bmatrix}$$

for an $(r-2) \times 2$ matrix B. Hence

$$M_1 M_2 \cdots M_s = (M_1 \cdots M_l)^{s/l} = \begin{bmatrix} I_2 & O \\ B(A^{s/l-1} + \cdots + A + I_2) & I_{r-2} \end{bmatrix}.$$

Since $|A| = 1$ and $A^{s/l} = I_2$, A does not have eigenvalues equal to 1, i.e., $|A - I_2| \neq 0$. Hence the lower left in the right of the above equation is equal to the zero matrix. □

Proposition 3. *If a* $\mathbf{Z}$*-weight ϕ satisfies (M-i) around a slice $\bar{\tau}$ in Δ_{in}^{r-3}, then the following holds. Let β_i, v_i be as in Lemma 1 and let $w_{i,k}$ be as in (M-i) (a). Then there exist elements $\mathbf{e}_1, \ldots, \mathbf{e}_s$ ($s = n(\bar{\tau}, r-1)$), $\mathbf{f}_1, \ldots, \mathbf{f}_{r-2}$ in N such that $\{\mathbf{e}_i, \mathbf{e}_{i+1}, \mathbf{f}_1, \ldots, \mathbf{f}_{r-2}\}$ is a basis of N and that*

$$\mathbf{e}_{i-1} + \phi(\beta_i, v_i)\mathbf{e}_i + \mathbf{e}_{i+1} + \sum_{k=1}^{r-2} \phi(\beta_i, w_{i,k})\mathbf{f}_k = 0 \tag{1}$$

holds for $1 \leq i \leq s$, where $\mathbf{e}_0 = \mathbf{e}_s$, $\mathbf{e}_{s+1} = \mathbf{e}_1$. Moreover,

$$\{\textit{faces of } \mathbf{R}_{\geq 0}\mathbf{e}_i + \mathbf{R}_{\geq 0}\mathbf{e}_{i+1} + \mathbf{R}_{\geq 0}\mathbf{f}_1 + \cdots + \mathbf{R}_{\geq 0}\mathbf{f}_{r-2} \mid 1 \leq i \leq s\} \tag{2}$$

is a fan in N.

Proof. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}_1, \ldots, \mathbf{f}_{r-2}\}$ be a basis of N. Let $\mathbf{e}_3$, $\mathbf{e}_4$, …, $\mathbf{e}_s$ be the elements in N determined by (1) for $2 \leq i \leq s-1$. Then (1) holds also for $i = s, 1$, because $(\mathbf{e}_{i-1}, \mathbf{e}_i, \mathbf{f}_1, \ldots, \mathbf{f}_{r-2})M_i = (\mathbf{e}_i, \mathbf{e}_{i+1}, \mathbf{f}_1, \ldots, \mathbf{f}_{r-2})$. Let $a_i = \phi(\beta_i, v_i)$ and let L be the submodule of N spanned by $\mathbf{f}_1, \ldots, \mathbf{f}_{r-2}$. Then

$\mathbf{e}_{i-1}+a_i\mathbf{e}_i+\mathbf{e}_{i+1}\in L$. If $a_i=-1$, then $\mathbf{e}_{i-2}+(a_{i-1}+1)\mathbf{e}_{i-1}+\mathbf{e}_{i+1}\in L$ and $\mathbf{e}_{i-1}+(a_{i+1}+1)\mathbf{e}_{i+1}+\mathbf{e}_{i+2}\in L$. Hence if $[[a_1,a_2,\ldots,a_s]]$ becomes $[[1,1,1]]$ by repeating the operation (*), then $\mathbf{e}_{i_1}+\mathbf{e}_{i_2}+\mathbf{e}_{i_3}\in L$ for certain integers $1\le i_1<i_2<i_3\le s$. While, if $[[a_1,a_2,\ldots,a_s]]$ becomes $[[b,0,-b,0]]$, then $\mathbf{e}_{i_1}+\mathbf{e}_{i_3}\in L$ and $\mathbf{e}_{i_4}+b\mathbf{e}_{i_1}+\mathbf{e}_{i_2}\in L$ for certain integers $1\le i_1<i_2<i_3<i_4\le s$. In both cases, $\{$faces of $\mathbf{R}_{\geq 0}q(\mathbf{e}_{i_j})+\mathbf{R}_{\geq 0}q(\mathbf{e}_{i_{j+1}})\mid 1\le j\le 3$ or $4\}$ is a fan in N/L, where $q: N\to N/L$ is the canonical projection. Hence so is $\{$faces of $\mathbf{R}_{\geq 0}q(\mathbf{e}_i)+\mathbf{R}_{\geq 0}q(\mathbf{e}_{i+1})\mid 1\le i\le s\}$. Therefore, (2) is a fan in N. □

Proposition 4. *If a* $\mathbf{Z}$*-weight* ϕ *satisfies (M-ii) around a slice* $\bar\tau$ *in* $\Delta^{r-3}_{\mathrm{in}}$*, then the following holds. There exist elements* $\mathbf{e}_i$ *in* $\mathbf{Z}^2$ *for all* i *in* $\mathbf{Z}$ *such that* $\{\mathbf{e}_i,\mathbf{e}_{i+1}\}$ *is a basis of* $\mathbf{Z}^2$ *and that*

$$\mathbf{e}_{i-1}+a_i\mathbf{e}_i+\mathbf{e}_{i+1}=0 \tag{3}$$

hold for all i *in* $\mathbf{Z}$*, where* $a_i=\phi(\beta_k,v_k)$*, if* $i\equiv k(\mathrm{mod}\ s)$*,* $1\le k\le s$*. Moreover, there exists an element* $\mathbf{e}^*$ *in* $\mathrm{Hom}(\mathbf{Z}^2,\mathbf{Z})$ *such that* $\langle \mathbf{e}^*,\mathbf{e}_i\rangle>0$ *for all* i *in* $\mathbf{Z}$*. This element* $\mathbf{e}^*$ *is unique except constant multiples, if and only if* $a'_1=\cdots=a'_l=-2$*.*

Proof. The first half is obvious. If $[[a_1,a_2,\ldots,a_s]]$ becomes $[[a'_1,a'_2,\ldots,a'_l]]$ by repeating the operation (*), then there exists a sequence of integers $\{i_j\}(j\in\mathbf{Z})$ such that $i_j<i_{j+1}$, that $i_j\equiv i_k(\mathrm{mod}\ s)$ if $j\equiv k(\mathrm{mod}\ l)$ and that $\mathbf{e}_{i_{j-1}}+a'_k\mathbf{e}_{i_j}+\mathbf{e}_{i_{j+1}}=0$, where $j\equiv k(\mathrm{mod}\ l)$, $1\le k\le l$. Let $\mathbf{e}^*$ be the element in $\mathrm{Hom}(\mathbf{Z}^2,\mathbf{Z})$ defined by $\langle\mathbf{e}^*,\mathbf{e}_{i_1}\rangle=\langle\mathbf{e}^*,\mathbf{e}_{i_2}\rangle=1$. Then

$$\langle\mathbf{e}^*,\mathbf{e}_{i_3}\rangle=-a'_2\langle\mathbf{e}^*,\mathbf{e}_{i_2}\rangle-\langle\mathbf{e}^*,\mathbf{e}_{i_1}\rangle\ge 2\langle\mathbf{e}^*,\mathbf{e}_{i_2}\rangle-\langle\mathbf{e}^*,\mathbf{e}_{i_2}\rangle=\langle\mathbf{e}^*,\mathbf{e}_{i_2}\rangle.$$

In a similar manner, we obtain $\langle\mathbf{e}^*,\mathbf{e}_{i_3}\rangle\le\langle\mathbf{e}^*,\mathbf{e}_{i_4}\rangle\le\cdots$, $\langle\mathbf{e}^*,\mathbf{e}_{i_1}\rangle\le\langle\mathbf{e}^*,\mathbf{e}_{i_0}\rangle\le\cdots$. If $a'_1=\cdots=a'_l=-2$, then the equalities hold in the above inequalities. Hence any element $\mathbf{f}^*$ in $\mathrm{Hom}(\mathbf{Z}^2,\mathbf{Z})$ with $\langle\mathbf{f}^*,\mathbf{e}_{i_k}\rangle>0$, is a constant multiple of $\mathbf{e}^*$. On the other hand, if $a'_{i_j}<-2$ for an integer j, then $\langle\mathbf{e}^*,\mathbf{e}_{i_{j+1}}\rangle>1$. The element $\mathbf{f}^*$ in $\mathrm{Hom}(\mathbf{Z}^2,\mathbf{Z})$ defined by $\langle\mathbf{f}^*,\mathbf{e}_{i_j}\rangle=\langle\mathbf{f}^*,\mathbf{e}_{i_{j+1}}\rangle=1$ is not a constant multiple of $\mathbf{e}^*$. □

Obviously, the following holds.

Lemma 5. *For each slice* $\bar\tau$ *in* $\Delta^{r-3}_{\mathrm{bd}}$ *with* $s:=n(\bar\tau,r-1)<\infty$*, there exist elements* $\tau_1,\ldots,\tau_s$ *in* A^{r-3}*,* β_1,β'_s *in* A^{r-2}_{bd}*,* $\beta'_1,\beta_2,\beta'_2,\ldots,\beta'_{s-1},\beta_s$ *in* A^{r-2}_{in} *satisfying the following:* $I^{-1}(\bar\tau)=\tau_1\cup\tau_2\cup\cdots\cup\tau_s$*,* $\tau_i\ne\tau_j$ *if* $i\ne j$*,* $A^{r-2}(\tau_i)=\{\beta_i,\beta'_i\}$ *for* $1\le i\le s$*,* $(\beta'_i,\beta_{i+1})\in B$ *for* $1\le i<s$*, and if* $s>1$*, then* $l_{(\beta'_i,\beta_{i+1})}(\tau_i)=\tau_{i+1}$ *for* $1\le i<s$*.*

We assume that A^{r-1} is finite, throughout the rest of this section. Hence $n(\bar\tau,r-1)<\infty$ for each slice $\bar\tau$ in $\Delta^{r-3}_{\mathrm{bd}}$. Now, we consider the following condition.

(C) For a slice $\bar\tau$ in $\Delta^{r-3}_{\mathrm{bd}}$, let τ_i, β_i, β'_i be as in the above lemma, let v_i be the vertices of β_i not contained in τ_i and let $a_i=\phi(\beta_i,v_i)$ for $2\le i\le s$. Then we obtain (C-i) or (C-ii) by repeating the following operation (**) on the sequence $[a_2,a_3,\ldots,a_s]$ of the above integers.

(**) Remove an integer a_i equal to -1 and add 1 to the integers $a_{i\pm1}$ on both sides of a_i, if $2<i<s$. While, if $i=2$ (resp. s) then add 1 only to the right (resp. left) integer a_3 (resp. a_{s-1}).

(C-i) $[a]$ $(a\le 0)$.

(C-ii) $[a'_1,a'_2,\ldots,a'_l]$ $(l\ge 2,\ a'_j\le -2)$.

Proposition 6. *If a* $\mathbf{Z}$*-weight* ϕ *satisfies (C) around a slice* $\bar\tau$ *in* $\Delta^{r-3}_{\mathrm{bd}}$*, then the following holds. Let* s*,* τ_i*,* β_i*,* β'_i*,* v_i *be as in the above lemma. If we determine elements* $\mathbf{e}_1,\ldots,\mathbf{e}_{s+1}$*,* $\mathbf{f}_1,\ldots,\mathbf{f}_{r-2}$ *in* N *so that* $\{\mathbf{e}_i,\mathbf{e}_{i+1},\mathbf{f}_1,\ldots,\mathbf{f}_{r-2}\}$ *is a basis of* N *for* $1\le i\le s$ *and that*

$$\mathbf{e}_{i-1}+\phi(\beta_i,v_i)\mathbf{e}_i+\mathbf{e}_{i+1}+\sum_{j=1}^{r-2}\phi(\beta_i,w_{i,j})\mathbf{f}_j=0$$

for $2 \le i \le s$, where $w_{i,j}$ are vertices of τ_i with $l_{(\beta'_i,\beta_{i+1})}(w_{i,j}) = w_{i+1,j}$ for $1 \le i < s$, $1 \le j \le r-2$, then there exists an element $\mathbf{e}^*$ in M such that $\langle \mathbf{e}^*, \mathbf{f}_j \rangle = \langle \mathbf{e}^*, \mathbf{e}_1 \rangle = \langle \mathbf{e}^*, \mathbf{e}_{s+1} \rangle = 0$, $\langle \mathbf{e}^*, \mathbf{e}_i \rangle > 0$ for $2 \le i \le s$ in the case (C-i) $a = 0$, or $\langle \mathbf{e}^*, \mathbf{f}_j \rangle = \langle \mathbf{e}^*, \mathbf{e}_1 \rangle = 0$, $\langle \mathbf{e}^*, \mathbf{e}_i \rangle > 0$ for $2 \le i \le s+1$ in the other cases.

Proof. Let $a_i = \phi(\beta_i, v_i)$ and let L be the submodule of N spanned by $\mathbf{f}_1, \ldots, \mathbf{f}_{r-2}$. If $[a_2, \ldots, a_s]$ becomes $[a]$ by repeating the operation (**), then $\mathbf{e}_1 + a\mathbf{e}_k + \mathbf{e}_{s+1} \in L$ for certain $2 \le k \le s$. Hence there exists an element $\mathbf{e}^*$ in M such that

$$\langle \mathbf{e}^*, \mathbf{f}_j \rangle = \langle \mathbf{e}^*, \mathbf{e}_1 \rangle = 0,\ \langle \mathbf{e}^*, \mathbf{e}_k \rangle = 1,\ \langle \mathbf{e}^*, \mathbf{e}_{s+1} \rangle = -a(\ge 0).$$

We easily see that $\langle \mathbf{e}^*, \mathbf{e}_i \rangle > 0$ for $2 \le i \le s$. Next, assume that $[a_2, \ldots, a_s]$ becomes $[a'_2, \ldots, a'_l]$ ($a'_j \le -2$). Then $\mathbf{e}_{i_{j-1}} + a'_j \mathbf{e}_{i_j} + \mathbf{e}_{i_{j+1}} \in L$ ($2 \le j \le l$) for certain $1 = i_1 < i_2 < \cdots < i_{l+1} = s+1$. Let $\mathbf{e}^*$ be the element in M determined by $\langle \mathbf{e}^*, \mathbf{f}_j \rangle = \langle \mathbf{e}^*, \mathbf{e}_1 \rangle = 0,\ \langle \mathbf{e}^*, \mathbf{e}_{i_2} \rangle = 1$. Then

$$\langle \mathbf{e}^*, \mathbf{e}_{i_3} \rangle = -a'_2 \langle \mathbf{e}^*, \mathbf{e}_{i_2} \rangle - \langle \mathbf{e}^*, \mathbf{e}_{i_1} \rangle > 2\langle \mathbf{e}^*, \mathbf{e}_{i_2} \rangle - \langle \mathbf{e}^*, \mathbf{e}_{i_2} \rangle = \langle \mathbf{e}^*, \mathbf{e}_{i_2} \rangle.$$

In a similar manner, we obtain $\langle \mathbf{e}^*, \mathbf{e}_{i_3} \rangle < \langle \mathbf{e}^*, \mathbf{e}_{i_4} \rangle < \cdots < \langle \mathbf{e}^*, \mathbf{e}_{i_{l+1}} \rangle$. Hence $\langle \mathbf{e}^*, \mathbf{e}_i \rangle \ge 1$ for $2 \le i \le s+1$. □

2 Construction of fans

Let $(A, B, \{l_b\}_{b \in B})$ be a finite simplicial system with a $\mathbf{Z}$-weight ϕ satisfying the conditions (M) and (C) around all slices in $\Delta^{r-3}_{\mathrm{in}}$ and $\Delta^{r-3}_{\mathrm{bd}}$, respectively. Let $\Delta^{r-3}_{\mathrm{f}}$ be the set of slices in $\Delta^{r-3}_{\mathrm{in}}$ around which ϕ satisfies (M-i) and let $\Delta^{r-3}_{\infty} = \Delta^{r-3}_{\mathrm{in}} \setminus \Delta^{r-3}_{\mathrm{f}}$. Let $T = \bigcup_{\alpha \in \Delta_{\mathrm{in}}} \alpha^{\mathrm{o}}$, where $\Delta_{\mathrm{in}} = \Delta^{r-1} \cup \Delta^{r-2}_{\mathrm{in}} \cup \Delta^{r-3}_{\mathrm{f}}$. Then T is a connected topological manifold by the condition (M-i) (a). First, we construct a simplicial system inducing a toplogical space $\widetilde{\Delta}$ with a Galois covering $\bar{f} : \widetilde{\Delta} \to \Delta$ whose restriction $\bar{f}_{|\bar{f}^{-1}(T)}$ to $\bar{f}^{-1}(T)$, is a universal covering.

For $b = (\beta_1, \beta_2)$ in B, we express as $b(1) = \beta_1$, $b(2) = \beta_2$, $\bar{b} = (\beta_2, \beta_1)$. Choose a simplex α_0 in A^{r-1}. Let

$$\mathrm{Path}(\Delta) = \Big(\{(b_1, \ldots, b_l) \mid b_i \in B, G(b_1(1)) = \alpha_0, G(b_i(2)) = G(b_{i+1}(1))\} \bigcup \{()\} \Big) / \sim .$$

Here $p \sim q$ implies that p turns to q by a finite repetition of the following two operations and their inverses.

1. Remove b_j, b_{j+1}, if $\overline{b_j} = b_{j+1}$.
2. Remove $b_{j+1}, b_{j+2}, \ldots, b_{j+s}$, if there exists a slice $\bar{\tau}$ in $\Delta^{r-3}_{\mathrm{f}}$ satisfying the following conditions. $I^{-1}(\bar{\tau}) = \tau_1 \cup \tau_2 \cup \cdots \cup \tau_s$, $A^{r-2}(\tau_i) = \{\beta_i, \beta'_i\}$, $b_{j+i} = (\beta'_i, \beta_{i+1})$ for $1 \le i \le s$, where $\beta_{s+1} = \beta_1$, $l_{b_{j+i}}(\tau_i) = \tau_{i+1}$ for $1 \le i \le s$, where $\tau_{s+1} = \tau_1$.

For each $p = [b_1, b_2, \ldots, b_l]$ in $\mathrm{Path}(\Delta)$, let $\alpha(p)$ be a copy of $\alpha_p := G(b_l(2))$ ($\alpha_{[\,]} = \alpha_0$) with an isomorphism $f_p : \alpha(p) \simeq \alpha_p$. Let $\widetilde{A}^{r-1} = \{\alpha(p) \mid p \in \mathrm{Path}(\Delta)\}$ and let $\widetilde{A} = \bigcup_{i=0}^{r-1} \widetilde{A}^i$, where $\widetilde{A}^i$ is the set of i-dimensional faces of simplices in $\widetilde{A}^{r-1}$. Let

$$\widetilde{B} = \{(f_p^{-1}(b_{l+1}(1)), f^{-1}_{[b_1,\ldots,b_l,b_{l+1}]}(b_{l+1}(2))) \mid p = [b_1, \ldots, b_l] \in \mathrm{Path}(\Delta), b_{l+1} \in B, G(b_{l+1}(1)) = \alpha_p\}.$$

Lemma 7. *The above $\widetilde{B}$ satisfies (i)-(iv) in the previous section.*

Proof. (i) and (iii) are obvious. Also (ii) holds, because

$$(f^{-1}_{[b_1,\ldots,b_{l+1}]}(b_{l+1}(2)), f^{-1}_{[b_1,\ldots,b_l]}(b_{l+1}(1)) = (f^{-1}_{[b_1,\ldots,b_{l+1}]}(\overline{b_{l+1}}(1)), f^{-1}_{[b_1,\ldots,b_{l+1},\overline{b_{l+1}}]}(\overline{b_{l+1}}(2))) \in \widetilde{B}.$$

For $\alpha([b_1,\dots,b_l])$ and $\alpha([b'_1,\dots,b'_m])$, $\beta'_1 = f^{-1}_{[b_1,\dots,b_l]}(b_l(2))$, $\beta_2 = f^{-1}_{[b_1,\dots,b_{l-1}]}(b_l(1))$, $\dots$, $\beta'_l = f^{-1}_{[b_1]}(b_1(2))$, $\beta_{l+1} = f^{-1}_{[]}(b_1(1))$, $\beta'_{l+1} = f^{-1}_{[]}(b'_1(1))$, $\dots$, $\beta_{l+m+1} = f^{-1}_{[b'_1,\dots,b'_m]}(b'_m(2))$ satisfy the condition (iv). $\square$

For $(\beta_1,\beta_2) \in \widetilde{B}$, let $l_{(\beta_1,\beta_2)} = f^{-1}_{\alpha_2} \circ l_b \circ f_{\alpha_1|\beta_1}$, where $\alpha_i = G(\beta_i)$, $b = (f_{\alpha_1}(\beta_1), f_{\alpha_2}(\beta_2)) \in B$. Then $\{l_b\}_{b\in\widetilde{B}}$ satisfies (v). Hence $(\widetilde{A}, \widetilde{B}, \{l_b\}_{b\in\widetilde{B}})$ is a simplicial system. Let $\widetilde{\Delta}$ be the induced topological space and let $\widetilde{I} : \bigcup_{\tilde{\alpha}\in\widetilde{A}} \tilde{\alpha} \to \widetilde{\Delta}$ be the quotient map. We can define the map $f : \sqcup_{\tilde{\alpha}\in\widetilde{A}}\tilde{\alpha} \to \sqcup_{\alpha\in A}\alpha$ by $\{f_p\}_{p\in\mathrm{Path}(\Delta)}$. Clearly, there exists a map $\bar{f} : \widetilde{\Delta} \to \Delta$ with $I \circ f = \bar{f} \circ \widetilde{I}$. Since B satisfies (iv), f and $\bar{f}$ are surjevtive. The restriction of $\bar{f}$ to $\bar{f}^{-1}(T)$ is unramified. Moreover, it is a universal covering, by the following.

Lemma 8. *$\bar{f}^{-1}(T)$ is simply connected.*

Proof. Let x_0 be a point in the interior of $\alpha([])$ and let $s : [0,1] \to \bar{f}^{-1}(T)$ be a continuous map with $s(0) = s(1) = \widetilde{I}(x_0)$. We may assume that s does not intersect with $\bar{f}^{-1}(\cup_{\alpha\in\Delta_{\mathrm{f}}^{r-3}}\alpha^{\mathrm{o}})$ and intersects with $E := \bar{f}^{-1}(\cup_{\alpha\in\Delta_{\mathrm{in}}^{r-2}}\alpha^{\mathrm{o}})$ transversally. Then $\mathrm{Im}(s) \cap E = \{s(t_1),\dots,s(t_l)\}$ for certain $0 = t_0 < t_1 < \cdots < t_l < t_{l+1} = 1$ and $(\bar{f}\circ s)([0,t_1) \cup (t_l,1]) \subset I(\alpha_0^{\mathrm{o}})$. There exist simplices α_i in A^{r-1} such that $(\bar{f}\circ s)((t_i,t_{i+1})) \subset I(\alpha_i^{\mathrm{o}})$ for $1 \le i \le l-1$. Let β_i be the $(r-2)$-dimensional face of α_i containing $\lim_{t\to t_i+0}(I^{-1}_{|\alpha_i^{\mathrm{o}}} \circ \bar{f}\circ s)(t)$ for $1 \le i \le l$, where $\alpha_l = \alpha_0$. Let β'_i be the $(r-2)$-dimensional face of α_i containing $\lim_{t\to t_{i+1}-0}(I^{-1}_{|\alpha_i^{\mathrm{o}}} \circ \bar{f}\circ s)(t)$ for $0 \le i \le l-1$. Then $b_i = (\beta'_{i-1}, \beta_i) \in B$ for $1 \le i \le l$ and $p = [b_1, b_2, \dots, b_l] \in \mathrm{Path}(\Delta)$. Since $s(t) \in \widetilde{I}(\alpha([b_1,\dots,b_i]))$ for $t_i < t < t_{i+1}$ and $s(1) \in \widetilde{I}(\alpha([]))$, $\alpha(p) = \alpha([])$, i.e., $p = []$. On the other hand, if $(b_1,\dots,b_l)$ turns to p' by one of the operations 1, 2 and their inverses, then there exists a continuous map $s' : [0,1] \to \bar{f}^{-1}(T)$ homotope to s from which we obtain p' by the above way. Hence s is zero homotope. $\square$

$\mathrm{Path}^0(\Delta) := \{p \in \mathrm{Path}(\Delta) \mid \alpha_p = \alpha_0\}$ is a group with respect to the product $[b_1,\dots,b_l][b'_1,\dots,b'_m] = [b_1,\dots,b_l,b'_1,\dots,b'_m]$ and acts on $\widetilde{A}^{r-1}$ freely, by $[b_1,\dots,b_l]\alpha([b'_1,\dots,b'_m]) = \alpha([b_1,\dots,b_l,b'_1,\dots,b'_m])$. $\mathrm{Path}^0(\Delta)$ acts also on $\bigcup_{\tilde{\alpha}\in\widetilde{A}}\tilde{\alpha}$ and $\widetilde{\Delta}$ by $px = (f^{-1}_{pq}\circ f_q)(x)$ for a point x in $\alpha(q)$. Hence $\bar{f} : \widetilde{\Delta} \to \Delta$ is a Galois covering and $\mathrm{Path}^0(\Delta)$ is the Galois group of $\bar{f}$.

Proposition 9. *There exist a map $h : \widetilde{\Delta}^0 (= \{\text{vetices of } \widetilde{\Delta}\}) \to N \setminus \{0\}$ and a homomorphism $\rho : \mathrm{Path}^0(\Delta) \to GL(N)$ satisfying the following: $\{(h\circ\widetilde{I})(v_1), (h\circ\widetilde{I})(v_2), \dots, (h\circ\widetilde{I})(v_r)\}$ is a basis of N for each simplex $\overline{v_1v_2\cdots v_r}$ in $\widetilde{A}^{r-1}$ and*

$$(h\circ\widetilde{I})(v_1) + (h\circ\widetilde{I})(w_1) + \sum_{i=2}^{r} \phi(f(\beta_2), f(w_i))(h\circ\widetilde{I})(w_i) = 0$$

holds for each $b = (\beta_1,\beta_2)$ in $\widetilde{B}$, where $\overline{v_1v_2\cdots v_r} = G(\beta_1)$, $\overline{w_1w_2\cdots w_r} = G(\beta_2)$ and $l_b(v_i) = w_i$ for $2 \le i \le r$. Moreover, $h(pv) = \rho(p)h(v)$ for $p \in \mathrm{Path}^0(\Delta)$ and $v \in \widetilde{\Delta}^0$.

Proof. Let u_1, u_2, $\cdots$, u_r be the vertices of $\alpha([\ ])$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r\}$ be a basis of N and let $H(u_i) = \mathbf{e}_i$ for $1 \le i \le r$. Next, assume that $\overline{u_2u_3\cdots u_r}$ is in A^{r-2}_{in} and that $b = (\overline{u_2u_3\cdots u_r}, \overline{w_2w_3\cdots w_r})$ is in $\widetilde{B}$ with $l_b(u_i) = w_i$ for $2 \le i \le r$. Let w_1 be the vertex of $G(\overline{w_2w_3\cdots w_r})$ not contained in $\overline{w_2w_3\cdots w_r}$. Let $H(w_i) = H(u_i)$ for $2 \le i \le r$ and define $H(w_1)$ by

$$H(u_1) + H(w_1) + \sum_{i=2}^{r} \phi(f(\overline{w_2w_3\cdots w_r}), f(w_i))H(w_i) = 0.$$

Then $\{H(w_1), H(w_2), \dots, H(w_r)\}$ is a basis of N. For any p in $\mathrm{Path}(\Delta)$, choosing a representative of p and repeating the above process, we can define $H(v_i)$ for the vertices v_i of $\alpha(p) = \overline{v_1v_2\cdots v_r}$. Then

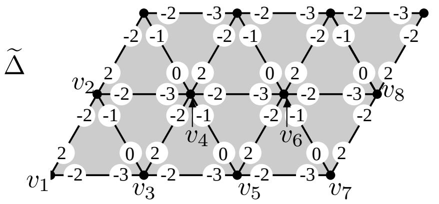
$\{H(v_1), H(v_2), \ldots, H(v_r)\}$ is a basis of N. Here, we note that $H(v_i)$ do not depend on the choice of a representaive of p, by Proposition 3. Moreover, $H(v) = H(w)$ if $\widetilde{I}(v) = \widetilde{I}(w)$ for v, w in $\widetilde{A}^0$. Hence there exists a map $h : \widetilde{\Delta}^0 \to N \setminus \{0\}$ with $H = h \circ \widetilde{I}_{|\widetilde{A}^0}$.

Recall that $pu_i = (f_p^{-1} \circ f_{[\ ]})(u_i)$ for p in $\mathrm{Path}^0(\Delta)$. Hence we can define the homomorphism $\rho(p)$: $\mathrm{Path}^0(\Delta) \to GL(N)$ by $\rho(p)H(u_i) = (H \circ f_p^{-1} \circ f_{[\ ]})(u_i)$. □

For any simplex $\overline{v_1 \cdots v_l}$ in $\widetilde{A}$, $\widetilde{I}(v_i) \neq \widetilde{I}(v_j)$ if $i \neq j$, by the above proposition. Hence any slice in $\widetilde{\Delta}$ is a simplex. Let

$$\Sigma(\Delta) = \{\mathbf{R}_{\geq 0} h(v_1) + \cdots + \mathbf{R}_{\geq 0} h(v_k) \mid \overline{v_1 \cdots v_k} \text{ are slices of } \widetilde{\Delta}\} \bigcup \{\{0\}\}.$$

Then $\Sigma(\Delta)$ consists of non-singular cones, by the above proposition. However, it may not be a fan. For example, Figure 2 is a part of $\widetilde{\Delta}$ for such Δ which is a triangulation with one vertex of a 2-dimensional real torus, where integers attached on both sides of edges on $\widetilde{\Delta}$ are pull-back of those on Δ. Since $h(v_8) = h(v_1) + h(v_3)$, $\Sigma(\Delta)$ is not a fan.



$$\begin{aligned}
h(v_4) &= -h(v_1) + h(v_2)\\
h(v_5) &= -2h(v_1) + h(v_2) - 2h(v_3)\\
h(v_6) &= -h(v_1) + h(v_2) - h(v_3)\\
h(v_7) &= 3h(v_1) - h(v_2) + 2h(v_3)\\
h(v_8) &= h(v_1) + h(v_3)
\end{aligned}$$

Figure 2

In the following, we consider the condition that $\Sigma(\Delta)$ becomes a fan. We define the map $\bar{h} : \widetilde{\Delta} \to S^{r-1} = (N_{\mathbf{R}} \setminus \{0\}) / \mathbf{R}_{>0}$ as follows. For each slice $\alpha = \overline{v_1 v_2 \cdots v_r}$ in $\widetilde{\Delta}^{r-1}$, let $\bar{h}_\alpha : \alpha \to S^{r-1}$ be the map defined by

$$\bar{h}_\alpha(t_1 v_1 + t_2 v_2 + \cdots + t_r v_r) = \mathrm{pr}(t_1 h(v_1) + t_2 h(v_2) + \cdots + t_r h(v_r)),$$

where $\mathrm{pr} : N_{\mathbf{R}} \setminus \{0\} \to S^{r-1}$ is the canonical projection. If $\alpha \cap \beta \neq \emptyset$ for α, β in $\widetilde{\Delta}^{r-1}$, then $\bar{h}_{\alpha|\alpha\cap\beta} = \bar{h}_{\beta|\alpha\cap\beta}$. Hence there exists a map $\bar{h} : \widetilde{\Delta} \to S^{r-1}$ with $\bar{h}_{|\alpha} = \bar{h}_\alpha$ for each α in $\widetilde{\Delta}^{r-1}$. If $\bar{h}$ is injective, then $\Sigma(\Delta)$ is a fan. The following holds, by Proposiion 3.

Proposition 10. *$\bar{h}_{|f^{-1}(T)}$ is a locally homeomorphism.*

For a point y in S^{r-1}, we denote by y^*, the symmetric point of y, i.e., $y^* = \mathrm{pr}(-v)$ if $y = \mathrm{pr}(v)$. Let

$$\overline{y_1 y_2} = \mathrm{pr}(\{(1-t)v_1 + tv_2 \mid 0 \leq t \leq 1\})$$

for $y_1 = \mathrm{pr}(v_1), y_2 = \mathrm{pr}(v_2) \neq y_1^*$ in S^{r-1}.

Definition. We call a subset X of S^{r-1} convex, if $\overline{xy} \subset X$ for any $x, y \neq x^*$ in X.

Definition. We call a slice $\bar{\tau}$ in $\Delta_{\mathrm{f}}^{r-3}$ symmetric, if $s = n(\bar{\tau}, r-1)$ is even and if $\phi(\beta_i, v_i) = \phi(\beta_{i+s/2}, v_{i+s/2})$ for $1 \leq i \leq s/2$, where τ_i, β_i are as in Lemma 1 and v_i are the vertices of β_i which are not contained in τ_i.

In Example 1, the unique edge in Δ_{f}^1 is symmetric, only if $\phi(\alpha_4, v_1) = \phi(\alpha_4, v_2) = 0$.

Proposition 11. *If a slice $\bar{\tau}$ in $\Delta_{\mathrm{f}}^{r-3}$ is symmetric, then $\mathbf{e}_i$, $\mathbf{e}_{i+s/2}$, $\mathbf{f}_1$, …, $\mathbf{f}_{r-2}$ are contained in an $(r-1)$-dimensional subspace of $N_{\mathbf{R}}$ for $1 \leq i \leq s/2$, where $\mathbf{e}_1$, …, $\mathbf{e}_s$ $\mathbf{f}_1$, …, $\mathbf{f}_{r-2}$ are as in Proposition 3.*

Proof. Let L be the submodule of N spanned by $\mathbf{f}_1$, …, $\mathbf{f}_{r-2}$ and let $q : N \to N/L (\simeq \mathbf{Z}^2)$ be the canonical projection. Then it suffices to show that $q(\mathbf{e}_{i+s/2}) = -q(\mathbf{e}_i)$ for $1 \leq i \leq s/2$. Let g

be the element in $GL(N/L)$ sending $q(\mathbf{e}_1)$ and $q(\mathbf{e}_2)$ to $q(\mathbf{e}_{1+s/2})$ and $q(\mathbf{e}_{2+s/2})$, respectively. Then $gq(\mathbf{e}_i) = q(\mathbf{e}_{[i+s/2]})$ for $3 \le i \le s$, where $[k] = k$ (resp. $k - s$), if $k \le s$ (resp. $> s$), because $q(\mathbf{e}_{i-1}) + a_i q(\mathbf{e}_i) + q(\mathbf{e}_{i+1}) = 0$ for $1 \le i \le s$ and $a_{i+s/2} = a_i$ for $1 \le i \le s/2$, where $a_i = \phi(\beta_i, v_i)$, $\mathbf{e}_0 = \mathbf{e}_s$, $\mathbf{e}_{s+1} = \mathbf{e}_1$. Hence $g^2 = 1$. Since $\mathbf{R}_{\ge 0} q(\mathbf{e}_i) + \mathbf{R}_{\ge 0} q(\mathbf{e}_{i+1}) \cap \mathbf{R}_{\ge 0} q(\mathbf{e}_{[i+s/2]}) + \mathbf{R}_{\ge 0} q(\mathbf{e}_{[i+s/2+1]}) = \{0\}$, $gx \ne x$ for any x in $N/L \setminus \{0\}$. Therefore, $gx = -x$. $\square$

Let $\tilde{\tau}$ be a slice in $\widetilde{\Delta}^{r-3}$ and assume that $\bar{f}(\tilde{\tau})$ is in $\Delta_{\mathrm{f}}^{r-3}$ and symmetric. Let $\widetilde{\Delta}^{r-2}(\tilde{\tau}) = \{\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_s\}$. Here, we may assume that $\tilde{\beta}_i$ and $\tilde{\beta}_{i+1}$ are faces of an $(r-1)$-dimensional simplex for each $1 \le i \le s$. Then $\bar{h}(\tilde{\beta}_i \cup \tilde{\beta}_{i+s/2})$ is contained in an $(r-2)$-dimensional great sphere of S^{r-1} for $1 \le i \le \frac{s}{2}$, by the above proposition.

Theorem 12. *If all slices in $\Delta_{\mathrm{f}}^{r-3}$ are symmetric, then $\bar{h}$ is injective and its image is convex.*

We borrow an idea in [8] for the proof.

Definition. We call an injective continuous map $s : [0,1] \to \widetilde{\Delta}$, segment, if the following holds: If $(\bar{h} \circ s)(1) \ne (\bar{h} \circ s)(0)^*$, then $\mathrm{Im}(\bar{h} \circ s) = \overline{(\bar{h} \circ s)(0)(\bar{h} \circ s)(1)}$. If $(\bar{h} \circ s)(1) = (\bar{h} \circ s)(0)^*$, then $(\bar{h} \circ s)(t) \ne (\bar{h} \circ s)(0)^*$ and $\mathrm{Im}(\bar{h} \circ s) = \overline{(\bar{h} \circ s)(0)(\bar{h} \circ s)(t)} \cup \overline{(\bar{h} \circ s)(t)(\bar{h} \circ s)(1)}$ for any t in $(0,1)$.

Proof of Theorem 12. It suffices to show that there exists a segment s with $s(0) = x_0$, $s(1) = x$ for any two points x_0, x in $\widetilde{\Delta}$ with $x_0 \ne x$. Fix a point x_0 in $\widetilde{\Delta}$ and choose a simplex α_0 in $\widetilde{\Delta}^{r-1}$ which contains x_0. Let $\Lambda_0 = \{\alpha_0\}$, $\Lambda_1 = \{\alpha \in \widetilde{\Delta}^{r-1} \mid \alpha_0 \cap \alpha \in \widetilde{\Delta}^{r-2}\}$, $\Lambda_{i+1} = \{\alpha \in \widetilde{\Delta}^{r-1} \setminus (\Lambda_{i-1} \cup \Lambda_i) \mid \alpha \cap \beta \in \widetilde{\Delta}^{r-2}$ for $^\exists \beta \in \Lambda_i\}$. Then $\Lambda_i \cap \Lambda_j = \emptyset$ if $i \ne j$ and $\widetilde{\Delta} = \bigcup_{i \ge 0} R_i$, where $R_i = \bigcup_{\alpha \in \Lambda_i} \alpha$. We show the following in the induction with respect to i.

For any point $x \ne x_0$ in R_i, there exists a segment s with $s(0) = x_0$, $s(1) = x$ satisfying the condition (A) : If $\mathrm{Im}(s) \cap \beta^{\mathrm{o}} = \{s(t_0)\}$ for $\beta \in \widetilde{\Delta}^{r-2}$, $0 < t_0 < 1$, then $s((t_1, t_0]) \subset R_j$, $s([t_0, t_2)) \subset R_{j+1}$ for certain $0 < t_1 < t_0 < t_2 < 1$, $0 \le j < i$.

For $i = 0$, the above holds, because $\bar{h}(\alpha_0)$ is convex. The value of the $\mathbf{Z}$-weight $\phi \le 0$, by the conditions (M), (C) and the assumption that all simplices in $\Delta_{\mathrm{f}}^{r-3}$ are symmetric. Hence if x is in a simplex α_1 in Λ_1, then $\overline{\bar{h}(x_0)\bar{h}(x)} \cap \bar{h}(\alpha_0 \cap \alpha_1) \ne \emptyset$. Therefore, the above holds also for $i = 1$. Now assume that $i \ge 2$, that the above holds for all j with $1 \le j \le i$ and that $x \in R_{i+1} \setminus R_i$. Let α be a simplex in Λ_{i+1} which contains x. Then there exists a simplex β_i in Λ_i with $\alpha \cap \beta_i \in \widetilde{\Delta}^{r-2}$. Let H be the closed hemisphere of S^{r-1} which contains $\bar{h}(\beta_i)$ and whose boundary contains $\bar{h}(\alpha \cap \beta_i)$, and let $y_0 = \bar{h}(x_0)$. Then y_0 is in H, by the condition (A) (see Figure 3). Assume that $\overline{y_0 \bar{h}(x)} \cap \bar{h}(\alpha \cap \beta_i) \ne \emptyset$. Then there exists a point x_1 on β_i such that $\overline{y_0 \bar{h}(x)} \cap \bar{h}(\alpha \cap \beta_i) = \{\bar{h}(x_1)\}$. Since there exists a segment s' with $s'(0) = x_0$, $s'(1) = x_1$, also does s with $s(0) = x_0$, $s(1) = x$ satisfying (A) (see Figure 4).

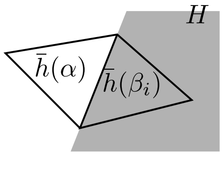


Figure 3

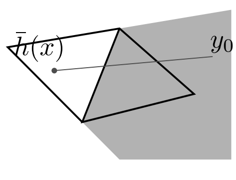


Figure 4

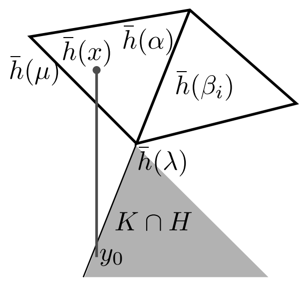


Figure 5

Now assume that $\overline{y_0 \bar{h}(x)} \cap \bar{h}(\alpha \cap \beta_i) = \emptyset$. Then there exists an $(r-2)$-dimensional face μ of α with $\overline{y_0 \bar{h}(x)} \cap \bar{h}(\mu) \ne \emptyset$ (see Figure 5). In the following, we show that there exists a simplex β_i' in Λ_i with

$\mu = \alpha \cap \beta_i'$. Since α is an $(r-1)$-dimensional simplex and μ, $\alpha \cap \beta_i$ are $(r-2)$-dimensional faces, $\lambda := \mu \cap (\alpha \cap \beta_i)$ is an $(r-3)$-dimensional simplex. λ is in $\widetilde{\Delta}_{\mathrm{f}}^{r-3}$, by Propositions 4 and 6. Hence there exist simplices β_{i-1}, β_{i-2}, …, β_j, β_{j+1}', …, β_i' in $\widetilde{\Delta}^{r-1}$ such that $\widetilde{\Delta}^{r-1}(\lambda) = \{\alpha, \beta_i, \beta_{i-1}, \dots, \beta_j, \beta_{j+1}', \dots, \beta_i'\}$, $\beta_k \cap \beta_{k+1}, \beta_k' \cap \beta_{k+1}' \in \widetilde{\Delta}^{r-2}(\lambda)$ for $j \le k < i$, $\beta_i' \cap \alpha = \mu$, where $j = i - \frac{1}{2}n(\lambda, r-1) + 1$, $\beta_j' = \beta_j$ (see Figure 6). Let K be the open hemisphere of S^{r-1} which does not contain $\bar{h}(\alpha)$ and whose boundary contains $\bar{h}(\mu)$. Then y_0 is in $K \cap H$. Moreover, $\bar{h}(\beta_j) \subset \overline{K} \cap H$, by Proposition 11. Let V be a small neighborhood of the image $\bar{h}(\mathrm{ct}(\lambda))$ under $\bar{h}$ of the center $\mathrm{ct}(\lambda)$ of λ. Then $V \cap \overline{K} \cap H \subset \bar{h}(\beta_j)$. Let w be a point in the interior β_i^{o} of β_i near $\mathrm{ct}(\lambda)$. Then $\overline{y_0 \bar{h}(w)} \cap \bar{h}(\beta_k \cap \beta_{k+1}) \neq \emptyset$ for $j \le k < i$. Hence $\beta_{i-1} \in \Lambda_{i-1}$, $\beta_{i-2} \in \Lambda_{i-2}$, …, $\beta_j \in \Lambda_j$, by the condition (A). Therefore, $\beta_{j+1}' \in \Lambda_{j+1}$, $\beta_{j+2}' \in \Lambda_{j+2}$, …, $\beta_i' \in \Lambda_i$. □

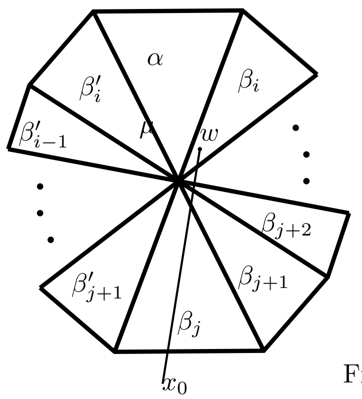
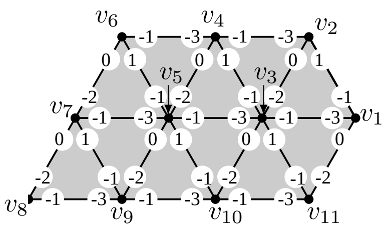


Figure 6

Remark 1. Even if a slice in $\Delta_{\mathrm{f}}^{r-3}$ is not symmetric, $\bar{h}$ may be injective. In Example 1, when $a \neq 0$, the unique edge in Δ_{f}^{1} is not symmetric. However, $\bar{h}$ is injective and $\Sigma(\Delta)$ is a fan. Let w_1, w_2, w_3, w_4 be the vertices of a tetrahedron in $\widetilde{\Delta}$ with $f(w_i) = v_i$ and let $\mathbf{e}_1 = h(w_1)$, $\mathbf{e}_2 = h(w_2) - h(w_1)$, $\mathbf{e}_3 = h(w_3)$, $\mathbf{e}_4 = h(w_4) - h(w_3)$. Then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is a basis of N and $\Sigma(\Delta) = \{\text{faces of } \mathbf{R}_{\ge 0}x_i + \mathbf{R}_{\ge 0}x_{i+1} + \mathbf{R}_{\ge 0}y_j + \mathbf{R}_{\ge 0}y_{j+1} \mid i, j \in \mathbf{Z}\}$, where

$$x_i = \mathbf{e}_1 + i\mathbf{e}_2 \quad \text{and} \quad y_j = \frac{1}{2}j(j-1)a\mathbf{e}_2 + \mathbf{e}_3 + j\mathbf{e}_4$$

2. Even if $\bar{h}$ is not injective, $\Sigma(\Delta)$ may be a fan. Figure 7 is a part of $\widetilde{\Delta}$ for such Δ which is a triangulation with one vertex of a 2-dimensional real torus. Let g_1 be the element in $\mathrm{Gal}(\bar{f})$ sending v_1, v_2, v_3 to v_9, v_7, v_8, respectively. Then $\rho(g_1) = I_3$. Let g_2 be the element in $\mathrm{Gal}(\bar{f})$ sending v_3, v_4, v_5 to v_1, v_2, v_3, respectively. Then $\mathrm{Gal}(\bar{f})$ is generated by g_1 and g_2.



$h(v_4) = -h(v_1) + 2h(v_3)$ $\qquad h(v_9) = h(v_1)$
$h(v_5) = h(v_1) - h(v_2) - h(v_3)$ $\qquad h(v_8) = h(v_3)$
$h(v_6) = 2h(v_1) - 2h(v_2) - 3h(v_3)$
$h(v_7) = h(v_2)$

Figure 7

We easily see that $h(v) + h(g_2 v) + h(g_2^2 v) = h(g_2^3 v)$ for any vertex v of $\widetilde{\Delta}$. Let $P = \overline{w_4 w_2 w_5}$, where $w_i = \bar{h}(v_i)$. Then $\rho(g_2)P \subset P$. Let $\tau = \overline{v_2 v_3 v_4} \cup \overline{v_3 v_4 v_5}$. Then τ is a fundamental domain of $\mathrm{Gal}(\bar{f})$ and $P \setminus \rho(g_2)P = \bar{h}(\tau)$ (see Figure 8). Hence $\widetilde{\Delta}/g_1^{\mathbf{Z}} \to S^2$ is injective and $\Sigma(\Delta)$ is a fan.

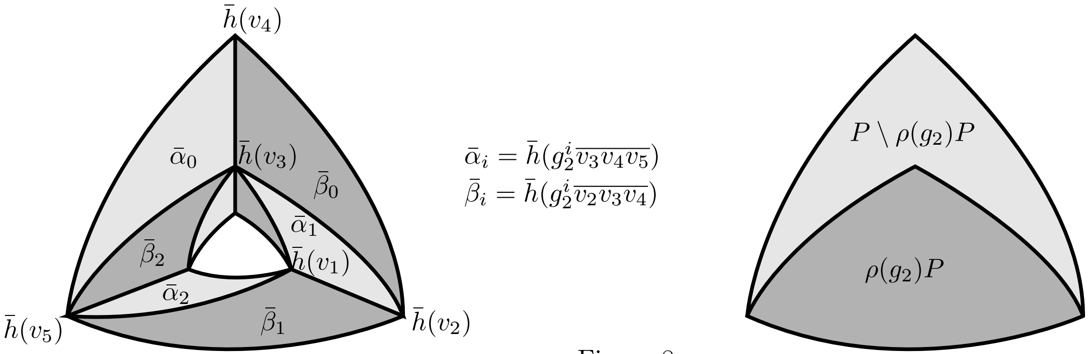


Figure 8

3 Subvarieties

Let $(A, B, \{l_b\}_{b\in B})$, ϕ and Δ be as in Section 1. Let $h : \widetilde{\Delta}^0 \to N$ and $\rho : \mathrm{Gal}(\bar{f}) \to GL(N)$ be the map and the homomorphism, respectively, in Proposition 9. Assume that $\bar{h} : \widetilde{\Delta} \to S^{r-1}$ is injective. Hence $\Sigma(\Delta) = \{\sigma(\tilde{\mu}) \mid \tilde{\mu}$ are simplices in $\widetilde{\Delta}\} \bigcup\{\{0\}\}$ is a fan, where $\sigma(\overline{v_1 \cdots v_l}) = \mathbf{R}_{\geq 0}h(v_1) + \cdots + \mathbf{R}_{\geq 0}h(v_l)$. Let $Y = T_N\mathrm{emb}(\Sigma(\Delta))$ be the toric variety associated to the fan $\Sigma(\Delta)$, and let $\Gamma = \rho(\mathrm{Gal}(\bar{f}))$ be the image of ρ. Then Y is non-singular and Γ acts on Y. Let $\tilde{\mu}$ be a simplex in $\widetilde{\Delta}$. We denote by $V(\tilde{\mu})$ the closure $\overline{\mathrm{orb}(\sigma(\tilde{\mu}))}$ of the orbit corresponding to the cone $\sigma(\tilde{\mu})$, which is an $(r - \dim\tilde{\mu} - 1)$-dimensional submanifold of Y. If $\dim\tilde{\mu} = r - 1$, then $V(\tilde{\mu})$ is a point. If $\bar{f}(\tilde{\mu})$ is in $\Delta^{r-2}_{\mathrm{in}}$ or $\Delta^{r-2}_{\mathrm{bd}}$, then $V(\tilde{\mu})$ is biholomorphic to $\mathbf{P}^1$ or $\mathbf{C}$, respectively. In the following, we study the structure of $V(\tilde{\mu})$ for a simplex $\tilde{\mu}$ with $\dim\tilde{\mu} \leq r-3$. For a simplex α in A and a proper face μ of α, we denote by α^μ, the maximal face of α with $\mu \cap \alpha^\mu = \emptyset$. Let $\bar{\mu}$ be a slice of Δ and assume that $s := \dim\bar{\mu} + 1 \leq r-2$. Let

$$A[\bar{\mu}]^{r-s-1} = \{\alpha^\mu \mid \mu \prec \alpha \in A^{r-1},\ \ I(\mu) = \bar{\mu}\},$$

and define $A[\bar{\mu}]^i$, $A[\bar{\mu}]$ in a similar manner as A^i, A. Here we assume that two different simplices in $A[\bar{\mu}]^{r-s-1}$ do not have common vertices. For example, in the case that $\bar{\mu} = I(v_1) = I(v_2)$ in Example 1, we consider v_3 in $\overline{v_1v_2v_3v_4}^{v_1} = \overline{v_2v_3v_4}$ and $\overline{v_1v_2v_3v_4}^{v_2} = \overline{v_1v_3v_4}$ as different points. Let

$$B[\bar{\mu}] = \{(\alpha^\nu, \beta^\mu) \mid (\alpha, \beta) \in B,\ \nu \prec \alpha, \mu \prec \beta,\ I(\nu) = I(\mu) = \bar{\mu},\ l_{(\alpha,\beta)}(\nu) = \mu\},$$

and let $l_{(\alpha^\nu,\beta^\mu)} = l_{(\alpha,\beta)|\alpha^\nu}$. Then $(A[\bar{\mu}], B[\bar{\mu}], \{l_b\}_{b\in B[\bar{\mu}]})$ is a simplicial system. Let $\Delta[\bar{\mu}]$ be the induced toplogical space and let $I_{\bar{\mu}} : \cup_{\alpha\in A[\bar{\mu}]}\alpha \to \Delta[\bar{\mu}]$ be the quotient map. We denote by $\phi_{[\bar{\mu}]}$ the $\mathbf{Z}$-weight of $(A[\bar{\mu}], B[\bar{\mu}], \{l_b\}_{b\in B[\bar{\mu}]})$ defined by $\phi_{[\bar{\mu}]}(\beta^\mu, v) = \phi(\beta, v)$. Obviously, $\phi_{[\bar{\mu}]}$ satisfies (vi) and the following holds.

Proposition 13. *Let $\bar{\mu}$ a slice of Δ with $s(= \dim\bar{\mu} + 1) \leq r-2$. If a slice $\bar{\beta}$ of Δ with $\bar{\mu} \prec \bar{\beta}$ is in $\Delta^{r-2}_{\mathrm{in}}$ or $\Delta^{r-3}_{\mathrm{in}}$, then $\bar{\beta}^{\bar{\mu}}$ is in $(\Delta[\bar{\mu}])^{r-s-2}_{\mathrm{in}}$ or $(\Delta[\bar{\mu}])^{r-s-3}_{\mathrm{in}}$, respectively, where $\bar{\beta}^{\bar{\mu}} = I_{\bar{\mu}}(\beta^\mu)$ for $\mu \prec \beta$ with $\bar{\mu} = I(\mu)$, $\bar{\beta} = I(\beta)$. If $r - s \geq 3$ and ϕ satisfies the condition (M) or (C) around a slice $\bar{\beta}$ in $\Delta^{r-3}(\bar{\mu})$, then so does $\phi_{[\bar{\mu}]}$ the same condition around $\bar{\beta}^{\bar{\mu}}$ in $(\Delta[\bar{\mu}])^{r-s-3}$. Moreover, if a slice $\bar{\beta}$ in $\Delta^{r-3}(\bar{\mu}) \cap \Delta^{r-3}_{\mathrm{f}}$, is symmetric, then so is $\bar{\beta}^{\bar{\mu}}$ in $(\Delta[\bar{\mu}])^{r-s-3}_{\mathrm{f}}$.*

Example 2. In Example 1, $\Delta[I(v_1)]$, $\Delta[I(v_3)]$, $\Delta[I(\overline{v_1v_3})]$, $\Delta[I(\overline{v_1v_2})]$ are as in Figure 9.

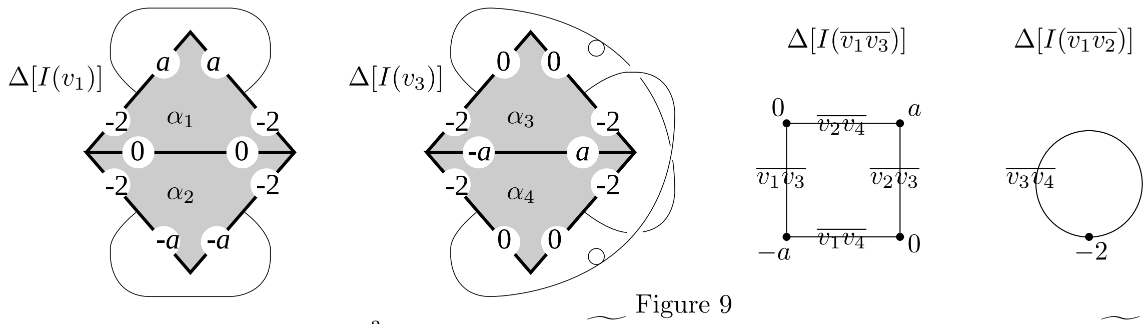


Figure 9

If $r-s=2$ and $\bar{\mu}$ is in $\Delta_{\mathrm{f}}^{r-3}$, then we define $\widetilde{\Delta[\bar{\mu}]}=\Delta[\bar{\mu}]$. In the other cases, we define $\widetilde{\Delta[\bar{\mu}]}$ in a similar manner as $\widetilde{\Delta}$. For slices $\tilde{\mu}$ of $\widetilde{\Delta}$, we can define $\widetilde{A}[\tilde{\mu}]$, $\widetilde{B}[\tilde{\mu}]$ and $\widetilde{\Delta}[\tilde{\mu}]$ in a similar manner as above.

Proposition 14. *Let $\bar{\mu}$ a slice of Δ with $s(=\dim\bar{\mu}+1)\leq r-2$ and let $\tilde{\mu}$ be a simplex of $\widetilde{\Delta}$ with $\bar{f}(\tilde{\mu})=\bar{\mu}$. If $r-s\geq 3$ and the map $h_{\bar{\mu}}:\widetilde{\Delta[\bar{\mu}]}^0\to N_{\bar{\mu}}:=\mathbf{Z}^{r-s}$ in Proposition 9 for the simplicial system $(A[\bar{\mu}],B[\bar{\mu}],\{l_b\}_{b\in B[\bar{\mu}]})$, is injective, then $\widetilde{\Delta[\bar{\mu}]}\simeq\widetilde{\Delta}[\tilde{\mu}]$ and $\mathrm{Path}(\Delta[\bar{\mu}])^0\simeq\left(\mathrm{Path}(\Delta)^0\right)_{\tilde{\mu}}:=\{p\in\mathrm{Path}(\Delta)^0\mid p\tilde{\mu}=\tilde{\mu}\}$.*

Proof. Choose an element μ in A with $I(\mu)=\bar{\mu}$ and take $G(\mu)$ and $G(\mu)^{\mu}$ as α_0 in the definition of $\mathrm{Path}(\Delta)$ and $\mathrm{Path}(\Delta[\bar{\mu}])$, respectively. Then we can define the map $i:\widetilde{A[\bar{\mu}]}^{r-s-1}\to\widetilde{A}[\tilde{\mu}]^{r-s-1}$ as

$$i(\alpha([(\beta_1'^{\mu},\beta_2^{\mu_2}),(\beta_2'^{\mu_2},\beta_3^{\mu_3}),\ldots,(\beta_l'^{\mu_l},\beta_{l+1}^{\mu_{l+1}})]))=\alpha([(\beta_1',\beta_2),(\beta_2',\beta_3),\ldots,(\beta_l',\beta_{l+1})])^{\nu},$$

by Proposition 13, where $\nu=f^{-1}_{[(\beta_1',\beta_2),\ldots,(\beta_l',\beta_{l+1})]}(\mu_{l+1})$. We easily see that this map is surjective. Hence i induces a surjective map $\bar{i}:\widetilde{\Delta[\bar{\mu}]}\to\widetilde{\Delta}[\tilde{\mu}]$. Let L be the submodule of N spanned by the images under h of the vertices of $\tilde{\mu}$. Let $h_{\tilde{\mu}}:\widetilde{\Delta}[\tilde{\mu}]^0\to N/L$ be the map induced by h and let $\widetilde{I}_{[\tilde{\mu}]}:\widetilde{A}[\tilde{\mu}]\to\widetilde{\Delta}[\tilde{\mu}]$ be the quotient map. Then

$$(h_{\tilde{\mu}}\circ\widetilde{I}_{[\tilde{\mu}]})(w_1)+(h_{\tilde{\mu}}\circ\widetilde{I}_{[\tilde{\mu}]})(w_1')+\sum_{i=2}^{r-s}\phi_{[\bar{\mu}]}(f(\overline{w_2\cdots w_{r-s}}),f(w_i))(h_{\tilde{\mu}}\circ\widetilde{I}_{[\tilde{\mu}]})(w_i)=0$$

for $\overline{w_1w_2\cdots w_{r-s}},\overline{w_1'w_2'\cdots w_{r-s}'}\in\widetilde{A}[\tilde{\mu}]^{r-s-1}$ with $\widetilde{I}_{[\tilde{\mu}]}(w_i)=\widetilde{I}_{[\tilde{\mu}]}(w_i')$ for $2\leq i\leq r-s$, because $\phi(f(\tilde{\beta}),f(w_i))=\phi_{[\bar{\mu}]}(f(\overline{w_2\cdots w_{r-s}}),f(w_i))$, where $\tilde{\beta}$ is the simplex in $\widetilde{A}^{r-2}$ with $\tilde{\beta}^{\tilde{\mu}'}=\overline{w_2\cdots w_{r-s}}$ for a face $\tilde{\mu}'$ of $\tilde{\beta}$. Let $g:N_{\bar{\mu}}\to N/L$ be the linear map sending $h_{\bar{\mu}}(u_i)$ to $(h_{\tilde{\mu}}\circ\bar{i})(u_i)$, where $u_1,\ldots,u_{r-s}$ are the vertices of $I_{[\bar{\mu}]}(\alpha_{[\bar{\mu}]}([]))$. Then $g\circ h_{\bar{\mu}}=h_{\tilde{\mu}}\circ\bar{i}_{|\widetilde{\Delta[\bar{\mu}]}^0}$. Hence $\bar{i}$ is injective and $\mathrm{Path}(\Delta[\bar{\mu}])^0\simeq\left(\mathrm{Path}(\Delta)^0\right)_{\tilde{\mu}}$. □

If $\bar{\mu}$ is in $\Delta_{\mathrm{f}}^{r-3}$ or $h_{\bar{\mu}}$ is injective, then $V(\tilde{\mu})$ is biholomorphic to the toric variety associated to the fan $\Sigma(\Delta[\bar{\mu}])$, by [4, Corollary 1.7] and the above proposition. In particular, $V(\tilde{\mu})$ is a compact toric surface, if $r-s=2$ and $\bar{\mu}$ is in $\Delta_{\mathrm{f}}^{r-3}$. For example, $V(\tilde{\mu})$ is a Hirzebruch surface for $\bar{\mu}=I(\overline{v_1v_3})$ in Example 1.

4 Quotients

We keep notations and assumptions in the previous section. Especially, $\bar{h}$ is injective. Let $\widetilde{X}=Y\setminus T_N$. Then $\widetilde{X}=\bigcup_{\tilde{\mu}\in\widetilde{\Delta}^0}V(\tilde{\mu})$. Let $\widetilde{\Delta}_{\mathrm{f}}^{r-3}=\{\tilde{\tau}\in\widetilde{\Delta}^{r-3}\mid\bar{f}(\tilde{\tau})\in\Delta_{\mathrm{f}}^{r-3}\}$ and let

$$\widetilde{\Delta}_{\mathrm{f}}^{i}=\{\tilde{\tau}\in\widetilde{\Delta}^{i}\mid\widetilde{\Delta}^{r-3}(\tilde{\tau})\subset\widetilde{\Delta}_{\mathrm{f}}^{r-3},\ |\widetilde{\Delta}^{r-1}(\tilde{\tau})|<\infty\}$$

for $0 \le i \le r-4$. Let

$$\widetilde{\Delta}_{\rm in} = \widetilde{\Delta}^{r-1} \bigcup \widetilde{\Delta}_{\rm in}^{r-2} \bigcup \bigcup_{i=0}^{r-3} \widetilde{\Delta}_{\rm f}^{i}$$

and let $\widetilde{Z} = \bigcup_{\tilde{\mu} \in \widetilde{\Delta}_{\rm in}} V(\tilde{\mu})$. Then $\widetilde{Z} \subset \widetilde{X}$. Let Γ be the image of the homomorphism $\rho : \mathrm{Path}^0(\Delta) \to GL(N)$ in Proposition 9 and let $\bar{h} : \widetilde{\Delta} \to S^{r-1}$ be the map in Section 2.

Proposition 15. *$D := \bar{h}(\bigcup_{\tilde{\tau} \in \widetilde{\Delta}_{\rm in}} \tilde{\tau}^{\rm o})$ is an open set of S^{r-1} and Γ acts on D properly discontinuously.*

Proof. Let $\tilde{\tau}$ be an element in $\widetilde{\Delta}_{\rm in}$. If $\dim \tilde{\tau} \ge r-3$, then a small neighborhood of any point in $\bar{h}(\tilde{\tau}^o)$ is contained in $\bar{h}(\bigcup_{\tilde{\alpha} \in \widetilde{\Delta}(\tilde{\tau})} \tilde{\alpha}^o)$, where $\widetilde{\Delta}(\tilde{\tau}) = \{\tilde{\tau}\} \cup \bigcup_{j=\dim \tilde{\tau}+1}^{r-1} \widetilde{\Delta}^j(\tilde{\tau})$. The same assertion holds also in the case that $\dim \tilde{\tau} \le r-4$, because $\widetilde{\Delta}_{\rm bd}^{r-2} \cap \widetilde{\Delta}^{r-2}(\tilde{\tau}) = \emptyset$ and $|\widetilde{\Delta}^{r-1}(\tilde{\tau})| < \infty$. Hence the first half holds. Let K be a compact set in D. Then $J = \{\tilde{\alpha} \in \widetilde{\Delta}^{r-1} \mid \bar{h}(\tilde{\alpha}) \cap K \neq \emptyset\}$ is a finite set. Since $\mathrm{Path}^0(\Delta)$ acts on $\widetilde{\Delta}^{r-1}$ freely, $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\} \subset \rho(\{p \in \mathrm{Path}^0(\Delta) \mid pJ \cap J \neq \emptyset\})$ is a finite set. $\square$

Let $C = \mathbf{R}_{>0} D$. Then C is an open cone, by the above proposition. Let $\mathrm{ord} : T_N \to N_{\mathbf{R}}$ be the map induced by $-\log|\ | : \mathbf{C}^\times \to \mathbf{R}$ and let

$$\widetilde{U} = \{y \in Y \mid \text{ there exists an open neighborhood } U_y \text{ of } y \text{ such that } U_y \setminus \widetilde{X} \subset \mathrm{ord}^{-1}(C)\}.$$

Here we note that if D is convex, then $\widetilde{U} = \mathrm{Int}(\overline{\mathrm{ord}^{-1}(C)})$. Obviously, $\widetilde{U} \cap T_N = \mathrm{ord}^{-1}(C)$. Let

$$C_{\tilde{\tau}} = \mathrm{Int}(\bigcup_{\tilde{\alpha} \in \widetilde{\Delta}^{r-1}(\tilde{\tau})} \sigma(\tilde{\alpha}))$$

and let

$$\widetilde{U}_{\tilde{\tau}} = \{y \in Y \mid \text{ there exists an open neighborhood } U_y \text{ of } y \text{ such that } U_y \setminus \widetilde{X} \subset \mathrm{ord}^{-1}(C_{\tilde{\tau}})\}$$

for $\tilde{\tau}$ in $\widetilde{\Delta}_{\rm in}$. Then $\widetilde{U} = \bigcup_{\tilde{\tau} \in \widetilde{\Delta}_{\rm in}} \widetilde{U}_{\tilde{\tau}}$ and $\widetilde{U}_{\tilde{\tau}} \supset V(\tilde{\tau})$. Hence $\widetilde{Z} \subset \widetilde{U}$.

Proposition 16. *Γ acts on $\widetilde{U}$ properly discontinuously.*

Proof. Since $\widetilde{U} \setminus \widetilde{X} = \mathrm{ord}^{-1}(C)$, if $\widetilde{U}_{\tilde{\tau}} \cap \widetilde{U}_{\tilde{\mu}} \neq \emptyset$, then $C_{\tilde{\tau}} \cap C_{\tilde{\mu}} \neq \emptyset$ for $\tilde{\tau}, \tilde{\mu}$ in $\widetilde{\Delta}_{\rm in}$. Hence $|\{\tilde{\mu} \in \widetilde{\Delta}_{\rm in} \mid \widetilde{U}_{\tilde{\tau}} \cap \widetilde{U}_{\tilde{\mu}} \neq \emptyset\}| < +\infty$ for any $\tilde{\tau}$ in $\widetilde{\Delta}_{\rm in}$. Since $\mathrm{Path}^0(\Delta)$ acts on $\widetilde{\Delta}^{r-1}$ freely, $|\{\gamma \in \Gamma \mid \gamma \widetilde{U}_{\tilde{\tau}} \cap \widetilde{U}_{\tilde{\mu}} \neq \emptyset\}| < +\infty$ for $\tilde{\tau}$, $\tilde{\mu}$ in $\widetilde{\Delta}_{\rm in}$. Hence Γ acts on $\widetilde{U}$ properly discontinuously. $\square$

Let $U = \widetilde{U}/\Gamma$, let $X = (\widetilde{X} \cap \widetilde{U})/\Gamma$ and let $Z = \widetilde{Z}/\Gamma$. Since $\widetilde{\Delta}$ is a dual graph of $\widetilde{X}$, so is Δ of X. If $\tilde{\tau}$ is in $\widetilde{\Delta}_{\rm in}$, then $V(\tilde{\tau})$ is compact. Hence we have:

Proposition 17. *Z is a compact analytic subset of U.*

5 Degenerating families

We keep notations and assumptions in the previous section. Let s be a positive integer smaller than r and let $L = \mathbf{Z}^s$. Let $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_s\}$ be a basis of L and let $\lambda = \mathbf{R}_{\ge 0}\mathbf{f}_1 + \mathbf{R}_{\ge 0}\mathbf{f}_2 + \cdots + \mathbf{R}_{\ge 0}\mathbf{f}_s$.

Proposition 18. *If there exists a continuous map $H : \Delta \to \lambda$ satisfying the following conditions (i), (ii), (iii), then there exists a linear map $F : N \to L$ sending all r-dimensional cones in $\Sigma(\Delta)$ to λ and $F \circ \gamma = F$ for all elements γ in Γ.*

(i) $H(v) \in L$ *for all vertices* v *of* Δ.
(ii) $\mathbf{R}_{\geq 0}H(\bar{\alpha}) = \lambda$ *for all* $(r-1)$*-dimensional slices* $\bar{\alpha}$ *of* Δ.
(iii) $\mathbf{R}_{\geq 0}(H \circ I)(\overline{v_2 \cdots v_r}) = \lambda$ *and*

$$(H \circ I)(v_1) + (H \circ I)(w_1) + \sum_{i=2}^{r} \phi(\overline{w_2 \cdots w_r}, w_i)(H \circ I)(w_i) = 0$$

for any $(\overline{v_2 \cdots v_r}, \overline{w_2 \cdots w_r})$ *in* B*, where* v_1 *and* w_1 *are the vertices of* $G(\overline{v_2 \cdots v_r})$ *and* $G(\overline{w_2 \cdots w_r})$ *not contained in* $\overline{v_2 \cdots v_r}$ *and* $\overline{w_2 \cdots w_r}$*, respectively.*

Proof. Choose a simplex $\overline{u_1 u_2 \cdots u_r}$ of $\widetilde{A}^{r-1}$. Since $\{(h \circ \widetilde{I})(u_1), (h \circ \widetilde{I})(u_2), \ldots, (h \circ \widetilde{I})(u_r)\}$ is a basis of N, we can define a linear map $F : N \to L$ so that $(F \circ h \circ \widetilde{I})(u_i) = (H \circ I \circ f)(u_i)$ for $1 \leq i \leq r$. If $(\overline{u_2 \cdots u_r}, \overline{w_2 \cdots w_r})$ is in $\widetilde{B}$, $l_{(\overline{u_2 \cdots u_r}, \overline{w_2 \cdots w_r})}(u_i) = w_i$ for $2 \leq i \leq r$ and $G(\overline{w_2 \cdots w_r}) = \overline{w_1 w_2 \cdots w_r}$, then by the condition (iii)

$$\begin{aligned}(F \circ h \circ \widetilde{I})(w_1) &= F(-(h \circ \widetilde{I})(u_1) - \sum_{i=2}^{r} \phi(f(\overline{w_2 \cdots w_r}), f(w_i))(h \circ \widetilde{I})(w_i)) \\ &= -(H \circ I \circ f)(u_1) - \sum_{i=2}^{r} \phi(f(\overline{w_2 \cdots w_r}), f(w_i))(H \circ I \circ f)(w_i) = (H \circ I \circ f)(w_1)\end{aligned}$$

because $\widetilde{I}(u_i) = \widetilde{I}(w_i)$ for $2 \leq i \leq r$. In a similar manner, we see that $(F \circ h \circ \widetilde{I})(w) = (H \circ I \circ f)(w)$ for all w in $\widetilde{A}^0$. Since $I \circ f = \bar{f} \circ \widetilde{I}$, $(F \circ h)(\tilde{w}) = (H \circ \bar{f})(\tilde{w})$ for all $\tilde{w}$ in $\widetilde{\Delta}^0$. Hence $(F \circ h)(g\tilde{w}) = (F \circ h)(\tilde{w})$ for any element g in $\mathrm{Gal}(\bar{f})$. □

F in the above proposition induces a holomorphic map $\varpi : U \to \mathbf{D}$, where $\mathbf{D} = \mathrm{Int}(\overline{[T_L \to L_{\mathbf{R}}]^{-1}(\lambda)})$ is an open set of $T_L\mathrm{emb}(\{\text{faces of } \lambda\}) \simeq \mathbf{C}^s$ biholomorphic to an s-dimensional polydisc. In the case $s = 1$, we obtain:

Corollary 19. *If* $\sum_{i=1}^{r-1} \phi(\beta, v_i) = -2$ *for any* $\beta = \overline{v_1 \cdots v_{r-1}}$ *in* A_{in}^{r-2}*, then there exists a holomorphic map* ϖ *from* U *to an open disc with* $\varpi^{-1}(0) = X$.

Obviously, the following holds.

Proposition 20. *If* Δ *and* ϕ *satisfy the assumption of the above proposition,* $\{\alpha \in \Delta_{\mathrm{bd}}^{r-2} \mid \mathbf{R}_{\geq 0}H(\alpha) = \lambda\} = \emptyset$ *and* $\{\alpha \in \Delta^i \mid \mathbf{R}_{\geq 0}H(\alpha) = \lambda\} = \Delta_{\mathrm{f}}^i$ *for* $s - 1 \leq i \leq r - 3$*, then* $\varpi^{-1}(\mathrm{orb}(\lambda)) = Z$.

Example 3. In Example 1, we can define $H : \Delta \to \lambda$ so that $(H \circ I)(v_1) = \mathbf{f}_1$, $(H \circ I)(v_3) = \mathbf{f}_2$ for $s = 2$. Then the assumption of Proposition 18 is satsified. $\varpi^{-1}(0)$ is irreducible and its normalization is biholomorphic to a Hirzebruch surface. Generic fibers of ϖ are biholomorphic to $(\mathbf{C}^{\times})^2/\langle g_1, g_2 \rangle$, which are hyperelliptic surfaces in the case $a = 0$, where $g_1 : (z_1, z_2) \mapsto (sz_1, tz_2^{-1})$, $g_2 : (z_1, z_2) \mapsto (z_1 z_2^a, tz_2)$ $(s, t \in \mathbf{C}^{\times})$.

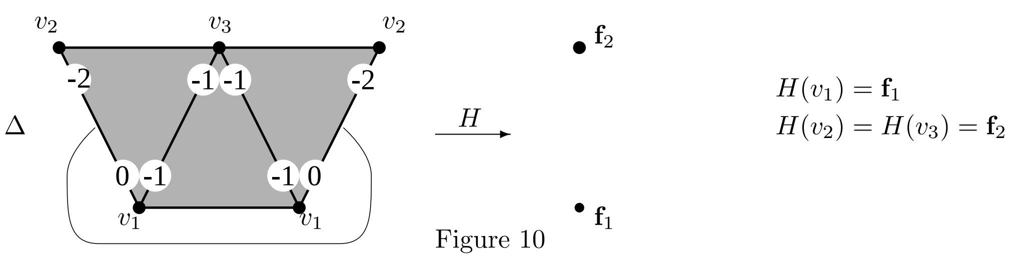


Figure 10

Example 4. Let $r = 3$, $s = 2$. Δ and H in Figure 10 satisfy the assumption of Proposition 18. Generic fibers of ϖ are elliptic curves. $\varpi^{-1}(\mathrm{orb}(\lambda))$ is a degenerate curve of type I_3 (see [2]). For $z \in \mathrm{orb}(\mathbf{R}_{\geq 0}\mathbf{f}_i)$, $\varpi^{-1}(z)$ are of type I_i. In a similar manner, we obtain degnerating families whose base spaces have any dimension greater than 2.

6 Singularities

We keep notations and assumptions in Sections 3 and 4. Moreover, we assume that all slices in $\Delta_{\mathrm{f}}^{r-3}$ are symmetric. Hence D in Proposition 15 is convex, by Theorem 12. Let $C' = |\Sigma(\Delta)|$. Then $C \subset C' \subset \overline{C}$.

Proposition 21. *Assume that for any $\beta = \overline{v_1 \cdots v_{r-1}}$ in A_{in}^{r-2},*

$$\sum_{i=1}^{r-1} \phi(\beta, v_i) \leq -2$$

and that Δ becomes a polyhedral decomposition, if we remove the images under I of all simplices in A_{in}^{r-2} on which the equality in the above inequality holds. Then C' is strongly convex.

Proof. Let $\square$ be the polyhedral decomposition in the proposition and let $\widetilde{\square}$ be the pull-back of $\square$ under $\bar{f} : \widetilde{\Delta} \to \Delta$. Let P be a polyhedron of $\widetilde{\square}$. By the assumption, there exists an element x_P in M such that $\langle x_P, h(w)\rangle = 1$ for any vertex w of $\widetilde{\Delta}$ contained in P. First, we show that $\langle x_P, h(w)\rangle > 1$ for any vertex w of $\widetilde{\Delta}$ not contained in P. Let $\tilde{h} : \widetilde{\Delta} \to N_{\mathbf{R}}$ be the piecewise linear map defined by $\tilde{h}(t_1v_1 + \cdots + t_rv_r) = t_1h(v_1) + \cdots + t_rh(v_r)$ on each simplex $\overline{v_1 \cdots v_r}$ of $\widetilde{\Delta}$. Let Q be a polyhedron of $\widetilde{\square}$ with $\dim(P \cap Q) = r - 2$. Then $\langle x_P, \tilde{h}(y)\rangle > 1$ for any point y in $Q \setminus (P \cap Q)$, by the assumption. Let $s : [0, 1] \to \widetilde{\Delta}$ be a segment in Section 2 such that $s(0)$ is in the interior of P and that $s(1) = w$. Then it suffices to show that $d(t) = \langle x_p, (\tilde{h} \circ s)(t)\rangle$ is an increasing function on $[t_0, 1]$, where t_0 is the real number with $s(t_0) \in \partial P$. We can take $s(0)$ so that $\mathrm{Im}(s)$ does not intersect simplices $\tilde{\tau}$ of $\widetilde{\Delta}$ with $\dim \tilde{\tau} \leq r - 3$. Let t_1, t_2 and t_3 be real numbers with $0 \leq t_1 < t_2 < t_3 \leq 1$. Then $(\tilde{h} \circ s)(t_2) = a(\tilde{h} \circ s)(t_1) + b(\tilde{h} \circ s)(t_3)$ for certain positive real numbers a and b. Let P_1 and P_3 be polyhedra of $\widetilde{\square}$ containing $s(t_1)$ and $s(t_3)$, respectively. Assume that $P_1 = P_3$. Then $a + b = 1$. Hence if $d(t_1) < d(t_2)$, then $d(t_2) < d(t_3)$. Assume that $\dim(P_1 \cap P_3) = r - 2$ and that $s(t_1), s(t_3) \notin P_1 \cap P_3$. Then $a + b < 1$, because there exists an element x in M such that $\langle x, \tilde{h}(y)\rangle = 1$ for all points y in P_1 and that $\langle x, \tilde{h}(y)\rangle > 1$ for all points y in $P_3 \setminus P_1$. Hence $d(t_2) = ad(t_1) + bd(t_3) \leq (a + b) \max\{d(t_1), d(t_3)\} < \max\{d(t_1), d(t_3)\}$. Therefore, $d(t)$ is an increasing function on $[t_0, 1]$.

Let F_{bd} (resp. F_{in}) be the set of $(r - 2)$-dimensional faces of P which are contained (resp. not contained) in the boundary of $\widetilde{\Delta}$. For any Q in F_{bd}, there exists an element x_Q in M such that $\langle x_Q, \tilde{h}(u)\rangle = 0$ (resp. ≥ 0) for any point u on Q (resp. P). Since C' is convex, $\langle x_Q, y\rangle \geq 0$ for any point y on C'. For any Q in F_{in}, there exists a polyhedron P' of $\widetilde{\square}$ with $P \cap P' = Q$. Let x_Q be the element in M such that $\langle x_Q, h(v)\rangle = 1$ for any vertex v of $\widetilde{\Delta}$ contained in P'. Then as we see in the above, $\langle x_Q, h(v)\rangle > 1$ for any vertex v of $\widetilde{\Delta}$ not contained in P'. Let $C_0 = \{y \in N_{\mathbf{R}} \mid \langle x_Q, y\rangle \geq 0 \text{ for } Q \in F_{\mathrm{bd}} \cup F_{\mathrm{in}}\}$. Then $C_0 \supset C'$ and $H_P \cap C_0$ is compact for the hyperplane H_P of $N_{\mathbf{R}}$ containing P. $\square$

Throughout the rest of this section, we assume that C' is strongly convex. Let $C^* = \{x \in M_{\mathbf{R}} \mid \langle x, y\rangle \geq 0 \text{ for } y \in C'\}$. Then C^* is an r-dimensional cone. For each x in M, we may consider the character $\mathbf{e}^x : T_N \to \mathbf{C}^{\times}$ of x as a rational function on Y, whose restriction to T_N is holomorphic. Here we use the notation $\mathbf{e}^x$ instead of $\mathbf{e}(x)$ in [4]. For a 1-dimensional cone σ in $\Sigma(\Delta)$, $\mathbf{e}^x$ has zero along $\mathrm{orb}(\sigma)$ of order $\langle x, y_\sigma\rangle$, where y_σ is the primitive element in N spaning σ. Hence $\mathbf{e}^x$ is a holomorphic function on Y for x in $C^* \cap M$. Moreover, for a cone σ in $\Sigma(\Delta)$, $\mathbf{e}^x$ does not vanish along $\mathrm{orb}(\sigma)$, if and only if $\langle x, y\rangle = 0$ for all points y in σ. Hence $\mathbf{e}^x$ vanishes along $\widetilde{Z}$ for any x in $C^* \cap M$, by Proposition 15.

Proposition 22. *$\sum_{x\in C^*\cap M}|\mathbf{e}^x|$ converges on $\widetilde{U}$.*

Proof. For any point z on $\widetilde{U}\setminus\widetilde{X}=\mathrm{ord}^{-1}(C)$, there exists an r-dimensional simplicial rational cone σ contained in C' and containing $\mathrm{ord}(z)$ in the interior. Since $\sigma^\vee=\{x\in M_{\mathbf{R}}\mid\langle x,y\rangle\geq 0 \text{ for } y\in\sigma\}\supset C^*$ and $\sum_{x\in\sigma^\vee\cap M}|\mathbf{e}^x|$ converges on $\mathrm{ord}^{-1}(\mathrm{Int}(\sigma))$ which contains z, so does $\sum_{x\in C^*\cap M}|\mathbf{e}^x|$. □

$|\mathbf{e}^{2x}|=\mathbf{e}^x\overline{\mathbf{e}^x}$ is a plurisubharmonic function on $\widetilde{U}$ vanishing along Z for any point x in $C^*\cap M$. $GL(N)$ acts on M to the right by $\langle x\gamma,y\rangle=\langle x,\gamma y\rangle$. Hence Γ acts on $C^*\cap M$. By the above proposition, $\sum_{x\in x_0\Gamma}|\mathbf{e}^{2x}|$ converges and is a Γ-invariant plurisubharmonic function on $\widetilde{U}$ for any point x_0 in $C^*\cap M$. We denote by $\mathbf{f}^{x_0}$ the induced function on U, i.e., $\mathbf{f}^{x_0}\circ[\widetilde{U}\to U]=\sum_{x\in x_0\Gamma}|\mathbf{e}^{2x}|$.

Let $\tilde{\tau}$ be a simplex in $\widetilde{\Delta}$ and let $C[\tilde{\tau}]=p(C)$ be the image of C under the projection $p:N_{\mathbf{R}}\to N_{\mathbf{R}}/\mathbf{R}\sigma(\tilde{\tau})$. By Proposition 15, $C[\tilde{\tau}]=N_{\mathbf{R}}/\mathbf{R}\sigma(\tilde{\tau})$, if and only if $\tilde{\tau}$ is in $\widetilde{\Delta}_{\mathrm{in}}$. Let $\Delta^i_{\mathrm{f}}=\{\bar{f}(\tilde{\tau})\mid\tilde{\tau}\in\widetilde{\Delta}^i_{\mathrm{f}}\}$ for $0\leq i\leq r-4$ and let

$$\Delta^i_{\mathrm{c}}=\{\bar{\tau}\in\Delta^i\mid C[\tilde{\tau}] \text{ is strongly convex for a simplex } \tilde{\tau} \text{ of } \widetilde{\Delta} \text{ with } \bar{f}(\tilde{\tau})=\bar{\tau}\}$$

for $0\leq i\leq r-3$. A slice $\bar{\tau}$ in Δ^{r-3}_∞ is not in $\Delta^{r-3}_{\mathrm{c}}$, if and only if $a'_1=a'_2=\cdots=a'_l=-2$ in the condition (M-ii), by Proposition 4. A slice $\bar{\tau}$ in $\Delta^{r-3}_{\mathrm{bd}}$ is not in $\Delta^{r-3}_{\mathrm{c}}$, if and only if (C-i) [0] holds in the condition (C), by Proposition 6. For $\bar{\tau}$ in Δ^i with $0\leq i\leq r-4$, if $\Delta[\bar{\tau}]$ satisfies the assumption of Proposition 21, then $\bar{\tau}$ is in Δ^i_{c}, by Propositions 13 and 14.

Theorem 23. *If $\Delta^i=\Delta^i_{\mathrm{f}}\cup\Delta^i_{\mathrm{c}}$ for $0\leq i\leq r-3$, then Z is contractible to a point in U.*

Proof. Let $\tilde{\tau}$ be a simplex in $\widetilde{\Delta}^{r-2}_{\mathrm{bd}}\bigcup\bigcup_{i=0}^{r-3}\widetilde{\Delta}^i_{\mathrm{c}}$ and let $s=\dim\sigma(\tilde{\tau})=\dim\tilde{\tau}+1$. Let $L=\{x\in M\mid\langle x,y\rangle=0 \text{ for } y\in\mathbf{R}\sigma(\tilde{\tau})\cap N\}$. Then $\dim(C^*\cap L_{\mathbf{R}})=\dim L=r-s$. Hence there exist linearly independent $r-s$ elements x_1, …, x_{r-s} in $C^*\cap L$. On the other hand, $C'\cap\mathbf{R}\sigma(\tilde{\tau})=\sigma(\tilde{\tau})$, because any face $\tilde{\mu}$ of $\tilde{\tau}$ is in $\widetilde{\Delta}^{\dim\tilde{\mu}}_{\mathrm{c}}$. Hence $p(C^*)=\sigma(\tilde{\tau})^\vee$, where $p:M_{\mathbf{R}}\to M_{\mathbf{R}}/L_{\mathbf{R}}$ is the canonical projection and $\sigma(\tilde{\tau})^\vee$ is the dual cone of $\sigma(\tilde{\tau})$ considered as a cone in $\mathbf{R}\sigma(\tilde{\tau})\cap N$. Since $\sigma(\tilde{\tau})$ is an s-dimensional non-singular cone, there exist s elements x'_{r-s+1}, …, x'_r in $p(C^*)$ such that $\{x'_{r-s+1},\ldots,x'_r\}$ is a basis of M/L and that $\mathbf{R}_{\geq 0}x'_{r-s+1}+\cdots+\mathbf{R}_{\geq 0}x'_r=p(C^*)$. There exists an element x_i in $C^*\cap M$ with $p(x_i)=x'_i$ for each $r-s+1\leq i\leq r$, because $\dim(C^*\cap L_{\mathbf{R}})=r-s$ and C^* is convex. Then $(\mathbf{e}^{x_1},\ldots,\mathbf{e}^{x_r}):Y(\tilde{\tau})\to(\mathbf{C}^\times)^{r-s}\times\mathbf{C}^s$ is a locally biholomorphic map, where $Y(\tilde{\tau})=T_N\mathrm{emb}(\{\text{faces of }\sigma(\tilde{\tau})\})\subset Y$. Hence $|\mathbf{e}^{2x_1}|+\cdots+|\mathbf{e}^{2x_r}|$ is strictly plurisubhrmonic on $Y(\tilde{\tau})$. Therefore, so is $\mathbf{f}^{x_1}+\cdots+\mathbf{f}^{x_r}$ on $\left(\bigcup_{\tilde{\mu}\in\Gamma\tilde{\tau}}Y(\tilde{\mu})\cap\widetilde{U}\right)/\Gamma$.

Since Δ consists of finetely many slices and the above $Y(\tilde{\tau})$ is an open neighborhood of $\mathrm{orb}(\sigma(\tilde{\tau}))$, there exists a finite subset S of $C^*\cap M$ such that $\mathbf{f}=\sum_{x\in S}\mathbf{f}^x$ is a plurisubharmonic function on U vanising only along Z and strictly plurisubharmonic on $U\setminus Z$. Since Z is compact, there exists a positive number ϵ such that the closure in U of the connected component U_ϵ of $\{z\in U\mid\mathbf{f}(z)<\epsilon\}$ containing Z, is compact. Then U_ϵ is strictly Levi pseudoconvex. Hence there exists a holomorphic map from U_ϵ to an analytic space sending Z to a point, by [1, XI, C, Theorem 4]. This map is biholomorphic on $U_\epsilon\setminus Z$, because the restriction $\mathbf{f}_{|U_\epsilon\setminus Z}$ of $\mathbf{f}$ is strictly plurisubharmonic. □

Assume that $\Delta^{r-2}_{\mathrm{bd}}=\emptyset$, that $\Delta^{r-3}=\Delta^{r-3}_{\mathrm{f}}$ and that Δ is a compact topological manifold. Then $\bar{f}:\widetilde{\Delta}\to\Delta$ is a universal covering, by Lemma 8. Hence $\Delta^i=\Delta^i_{\mathrm{f}}$ for $0\leq i\leq r-4$ and $X=Z$. Thus we obtain a cusp singularity, by the above theorem,

Example 5. The left Δ in Figure 11 satisfies the condition (M) and the assumption of Proposition 21. Z is irreducible and its normalization is biholomorphic to a toric surface obtained from $\mathbf{P}^1\times\mathbf{P}^1$ blowing up 4 points(see the right in Figure 11).

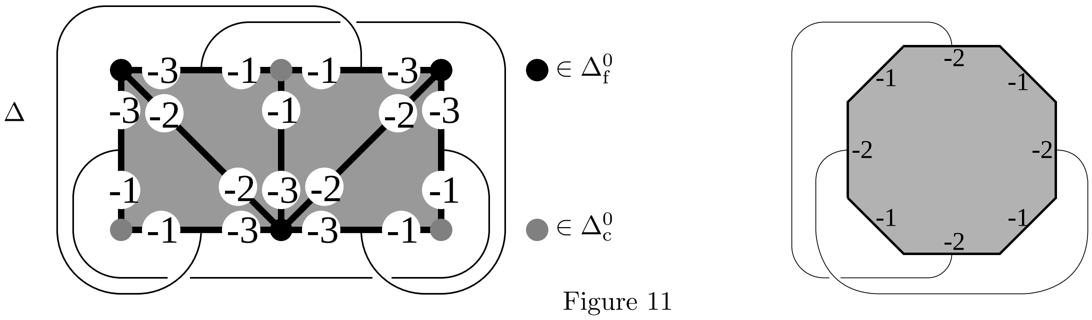


Figure 11

Example 6. There exists an r-dimensional isolated singularity with a resolution whose exceptional set is a curve satisfying the following conditions (1) and (2).

(1) Each irreducible component is a non-singular rational curve with only two points at which other irreducible components intersect, or a rational curve with a node only at which other irreducible components intersect.

(2) At any singular point, at most r branches of irreducible components intersect to each other.
A simplicial system giving the above, is obtained as follows: To each singular point, correspond an $(r-1)$-dimensional simplex and if there exists an irreducible curve passing through singular points x and y, then glue $(r-2)$-dimensional faces of the simplices corresponding to x and y. Next determine a $\mathbf{Z}$-weight ϕ so that ϕ satisfies (M-ii) and (C) around slices in Δ_{in}^{r-3} and Δ_{bd}^{r-3}, respectively. This condition is satisfied, if $\phi \leq -3$. For example, the right curve in Figure 12 is obtained from the left Δ. Also a curve consisting of rational curves intersecting to each other as edges of a polyhedron, satisfies the above condition (1).

Figure 12

Theorem 24. *If (I) or (II) in the following holds for each simplex $\tilde{\tau}$ in $\widetilde{\Delta}_{\text{in}}$, then there exists a surjective proper holomorphic map $\pi : V \to W$ from an open neighborhood V of Z to a finite covering W of a Stein analytic space such that $\pi^* : \mathcal{O}_W \to \mathcal{O}_V$ is an isomorphism, that the restriction $\pi_{|V \setminus X}$ of π to $V \setminus X$ is biholomorphic and that each fiber of π is connected, where $\mathcal{O}_V$ and $\mathcal{O}_W$ are the rings of the holomorphic functions on V and W, respectively.*

(I) If all faces of $\tilde{\tau}$ are in $\widetilde{\Delta}_{\text{in}}$, then there exists an element x_0 in $C^* \cap M$ such that $\langle x, h(v)\rangle \geq \langle x_0, h(v)\rangle$ for all x in $x_0\Gamma$ and all vertices v of $\tilde{\tau}$ and that for any x in $x_0\Gamma \setminus \{x_0\}$, there exists a vertex v of $\tilde{\tau}$ with $\langle x, h(v)\rangle > \langle x_0, h(v)\rangle$.

(II) If there exists a face of $\tilde{\tau}$ which is not in $\widetilde{\Delta}_{\text{in}}$, then for each face $\tilde{\mu}$ of $\tilde{\tau}$ which is not in $\widetilde{\Delta}_{\text{in}}$, there exists an element $x_{\tilde{\mu}}$ in $C^* \cap M$ satisfying the following two conditions (1) and (2).

(1) $\langle x_{\tilde{\mu}}, h(v)\rangle = 0$ for all vertices v of $\tilde{\mu}$.

(2) $\langle x, h(v)\rangle \geq \langle x_{\tilde{\mu}}, h(v)\rangle$ for all x in $x_{\tilde{\mu}}\Gamma$ and all vertices v of $\tilde{\tau}$, and for any x in $x_{\tilde{\mu}}\Gamma \setminus \{x_{\tilde{\mu}}\}$, there exists a vertex v of $\tilde{\tau}$ with $\langle x, h(v)\rangle > \langle x_{\tilde{\mu}}, h(v)\rangle$.

Proof. First, we show that there exist an open neighborhood V_0 of Z relatively compact in U and holomorphic functions $f_1, \ldots, f_s$ on U such that $\{w \in V_0 \mid f_1(w) = \cdots = f_s(w) = 0\} = Z$. We note that $\widetilde{Z} = \sqcup_{\tilde{\tau} \in \widetilde{\Delta}_{\text{in}}} \text{orb}(\sigma(\tilde{\tau}))$, because the closure of $\text{orb}(\sigma(\tilde{\tau}))$ is the union of $\text{orb}(\sigma(\tilde{\mu}))$ with $\tilde{\tau} \prec \tilde{\mu}$ and if $\tilde{\tau} \in \widetilde{\Delta}_{\text{in}}$ and $\tilde{\tau} \prec \tilde{\mu}$, then $\tilde{\mu} \in \widetilde{\Delta}_{\text{in}}$. Let $\tilde{\tau}$ be a simplex in $\widetilde{\Delta}_{\text{in}}$. If (I) holds for $\tilde{\tau}$, then for any point z in orb$(\sigma(\tilde{\tau}))$, there exists a neighborhood $\widetilde{U}_z$ of z such that

$$\{w \in \widetilde{U}_z \mid \tilde{f}^{x_0}(w) = 0\} = \{w \in \widetilde{U}_z \mid \mathbf{e}^{x_0}(w) = 0\} = (V(v_1) \cup \cdots \cup V(v_l)) \cap \widetilde{U}_z = \widetilde{Z} \cap \widetilde{U}_z,$$

where $\tilde{f}^{x_0} = \sum_{x \in x_0 \Gamma} \mathbf{e}^x$ and $v_1, \ldots, v_l$ are the vertices of $\tilde{\tau}$. Assume that (II) holds for $\tilde{\tau}$. Then for any point z in orb$(\sigma(\tilde{\tau}))$ and for each face $\tilde{\mu}$ of $\tilde{\tau}$ not in $\widetilde{\Delta}_{\mathrm{in}}$, there exists a neighborhood $\widetilde{U}_{z,\tilde{\mu}}$ of z such that $\{w \in \widetilde{U}_{z,\tilde{\mu}} \mid \tilde{f}^{x_{\tilde{\mu}}}(w) = 0\} = \{w \in \widetilde{U}_{z,\tilde{\mu}} \mid \mathbf{e}^{x_{\tilde{\mu}}}(w) = 0\}$, by the condition (2), where $\tilde{f}^{x_{\tilde{\mu}}} = \sum_{x \in x_{\tilde{\mu}} \Gamma} \mathbf{e}^x$. Hence $\tilde{f}^{x_{\tilde{\mu}}}$ does not vanish on $\widetilde{U}_{z,\tilde{\mu}} \setminus (V(v_1) \cup \cdots \cup V(v_l))$ by the condiiton (1), where $v_1, \ldots, v_l$ are the vertices of $\tilde{\tau}$ which are not on $\tilde{\mu}$. Therefore, $\{w \in \widetilde{U}_z \mid \tilde{f}^{x_{\tilde{\mu}}}(w) = 0 \text{ for all } \tilde{\mu} \in \Lambda\} = \widetilde{Z} \cap \widetilde{U}_z$, where Λ is the set of faces of $\tilde{\tau}$ not in $\widetilde{\Delta}_{\mathrm{in}}$ and $\widetilde{U}_z = \bigcap_{\tilde{\mu} \in \Lambda} \widetilde{U}_{z,\tilde{\mu}}$. Since Z is compact, there exist finitely many points $z_1, \ldots, z_l$ on $\widetilde{Z}$ such that $V_0 := U_{z_1} \cup \cdots \cup U_{z_l}$ is an open neighborhood of Z relatively compact in U, where U_{z_i} are the images of $\widetilde{U}_{z_i}$ under the canonical projection $\widetilde{U} \to U$. Moreover, there exist holomorphic functions $f_1, \ldots, f_s$ on U with $\{w \in V_0 \mid f_1(w) = \cdots = f_s(w) = 0\} = Z$, because the above functions $\tilde{f}^{x_0}$ and $\tilde{f}^{x_{\tilde{\mu}}}$ are Γ-invariant.

Let $\mathbf{f}_0 = |f_1| + |f_2| + \cdots + |f_s|$. Then $\{z \in V_0 \mid \mathbf{f}_0(z) = 0\} = Z$. Since ∂V_0 is compact and $Z \cap \partial V_0 = \emptyset$, $\epsilon_0 := \inf\{\mathbf{f}_0(z) \mid z \in \partial V_0\} > 0$. Hence $V := \{z \in V_0 \mid |f_i(z)| < \epsilon_0/s \text{ for } 1 \le i \le s\}$ is an open neighborhood of Z relatively compact in U and for any point z in ∂V there exists a suffix i with $|f_i(z)| = \epsilon_0/s$. Let K be a compact set in V. Then $m_i := ||f_i||_K = \sup\{|f_i(z)| \mid z \in K\} < \epsilon_0/s$. Hence $\{z \in V \mid |f(z)| \le ||f||_K \text{ for all } f \in \mathcal{O}_V\} \subset \{z \in V \mid |f_i(z)| \le m_i \text{ for } 1 \le i \le s\}$ is compact. Thus by [1, VII, D, Theorem 9], there exists a surjective holomorphic map $\pi_0 : V \to W_0$ from V to a Stein analytic space W_0 such that $\pi_0^* : \mathcal{O}_{W_0} \to \mathcal{O}_V$ is an isomorphism. Let K' be a compact set in W_0. Then $m_i' := ||(\pi_0^*)^{-1}(f_i)||_{K'} < \epsilon_0/s$. Hence $\pi_0^{-1}(K') \subset \{z \in V \mid |f_i(z)| \le m_i' \text{ for } 1 \le i \le s\}$ is compact. Let $u_1, u_2, \ldots, u_r$ be linearly independent r elements in $C^* \cap M$. Then $|\mathbf{e}^{2u_1}| + |\mathbf{e}^{2u_2}| + \cdots + |\mathbf{e}^{2u_r}|$ is a plurisubharmonic function on Y whose restriction to $Y \setminus \widetilde{X}$ is strictly plurisubharmonic. Hence $\mathbf{f}^{u_1} + \mathbf{f}^{u_2} + \cdots + \mathbf{f}^{u_r}$ is a plurisubharmonic function on U whose restriction to $U \setminus X$ is strictly plurisubharmonic. Therefore, $\pi_0^{-1}(\pi_0(z))$ consists of finite points, because it is compact for any point z in $V \setminus X$. Applying Stein factorization(see [7, Theorem 1.9]) to π_0, we obtain a desired map π. □

The holomorphic map π in the above theorem maps Z to a point. On the other hand, irreducible components of X which are not contained in Z, may contract to non-isolated singularities. In the following, we give two examples of simplicial systems which give 3-dimensional non-isolated singularities.

Example 7. Let Δ be the triangulation of a compact topological surface obtained by gluing edges joined by a thin curved line together of the development in Figure 13. Obviously, Δ is induced from a simplicial system and $\Delta^0 = \Delta^0_\infty$. Let $\overline{v_1 v_2 v_3}$ be a triangle of $\widetilde{\Delta}$ and let $\mathbf{e}_i = h(v_i)$ for $i = 1, 2, 3$. The faces of $\overline{v_1 v_2 v_3}$ not in $\widetilde{\Delta}_{\mathrm{in}}$ are the vertices v_1, v_2 and v_3. Let $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$ be the basis of M dual to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. In the following, we show that $x_{\tilde{\mu}} = \mathbf{e}_2^* + \mathbf{e}_3^*$ satisfies the condtion (II) for $\tilde{\tau} = \overline{v_1 v_2 v_3}$ or $\overline{v_1 v_2}$ and $\tilde{\mu} = v_1$. Considering the canonical projection $N \to N/\mathbf{Z}\mathbf{e}_1$ and by Proposition 4, we see that $\{x \in C^* \cap M \mid \langle x, \mathbf{e}_1 \rangle = 0\} = \mathbf{Z}_{\ge 0} x_{\tilde{\mu}}$. Suppose that $\langle x_{\tilde{\mu}} \gamma, \mathbf{e}_2 \rangle = 0$ for an element γ in Γ. Then $\langle x_{\tilde{\mu}}, \gamma \mathbf{e}_2 \rangle = 0$. However, $\{v \in \widetilde{\Delta}^0 \mid \langle x_{\tilde{\mu}}, h(v) \rangle = 0\} = \{v_1\}$ and there does not exist an element p in Path$^0(\Delta)$ with $pv_2 = v_1$, because $\bar{f}(v_1) \neq \bar{f}(v_2)$. Hence $\langle x, \mathbf{e}_2 \rangle \ge 1 = \langle x_{\tilde{\mu}}, \mathbf{e}_2 \rangle$ for any element x in $x_{\tilde{\mu}}\Gamma$. Therefore, $x_{\tilde{\mu}}$ satisfies the condtions (1) and (2) in (II). Now, we describe the structure of X and the restriction to $X \cap V$ of the holomorphic map π in the above theorem. Z consists of three rational curves meeting at two points (see Figure 14). Let $\widetilde{X}_i = \mathrm{orb}(\mathbf{R}_{\ge 0} \mathbf{e}_i) \cap \widetilde{U}$ and let X_i be the image of $\widetilde{X}_i$ under the canonical projection $\widetilde{U} \to U$ for $i = 1, 2, 3$. Then $U = (U \setminus X) \sqcup X_1 \sqcup X_2 \sqcup X_3 \sqcup Z$ and $X_i \simeq \widetilde{X}_i / \Gamma_{\mathbf{e}_i}$ are elliptic surfaces. Let $x_i = \mathbf{e}_1^* + \mathbf{e}_2^* + \mathbf{e}_3^* - \mathbf{e}_i^*$. Then x_i is a $\Gamma_{\mathbf{e}_i}$-invariant element in $C^* \cap M$, because $\langle x_i, \gamma \mathbf{e}_i \rangle = 0$ and $\langle x_i, \gamma \mathbf{e}_j \rangle = 1$ for any element γ in $\Gamma_{\mathbf{e}_i}$ and $j \neq i$, and $\mathbf{e}^x$ vanishes along $\widetilde{X}_i$ for any element x in $x_i \Gamma \setminus \{x_i\}$. Hence the holomorphic function f_i on U with $f_i \circ [\widetilde{U} \to U] = \sum_{x \in x_i \Gamma} \mathbf{e}^x$, gives a fibration on X_i.

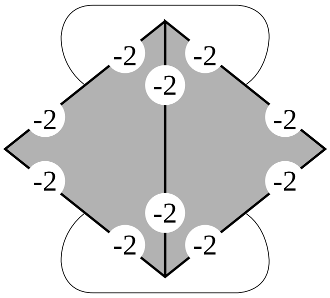


Figure 13

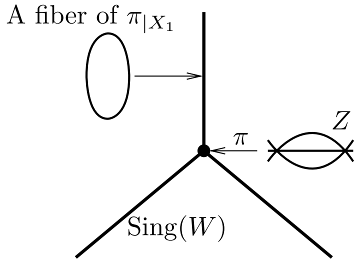


Figure 14

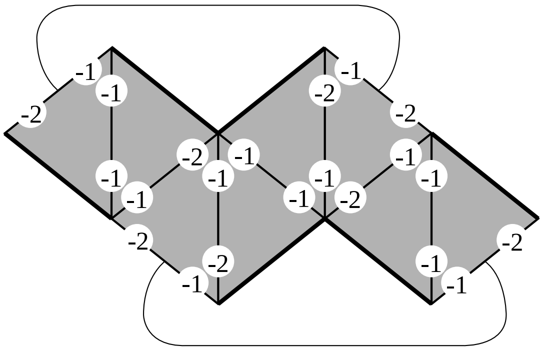


Figure 15

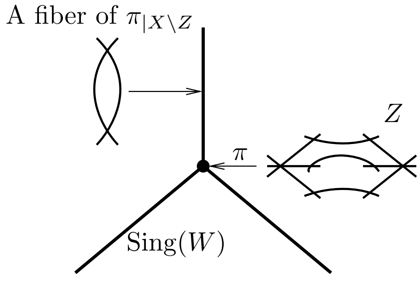


Figure 16

Example 8. Let Δ be the triangulation of a 2-dimensional compact topological space obtained by gluing edges joined by a thin curved line together of the development in Figure 15. Obviously, Δ is induced from a simplicial system and all vertices in $\Delta^0 = \Delta^0_{\mathrm{bd}}$ satisfy the condition (C-i) with $a = 0$. We can easily verify the following two statements. For all vertices v of $\widetilde{\Delta}$, there exist primitive elements x_v in $C^* \cap M$ such that $\{x \in C^* \cap M \mid \langle x, h(v)\rangle = 0\} = \mathbf{Z}_{\geq 0} x_v$ and that $\langle x_v, h(w)\rangle = 1$ (resp. 0) for all vertices w of $\widetilde{\Delta}$ which and v are joined by an edge in $\widetilde{\Delta}_{\mathrm{in}}$ (resp. $\widetilde{\Delta}^1_{\mathrm{bd}}$). If $\langle x_v, h(w)\rangle = 0$ for a vertex w of $\widetilde{\Delta}$, then $\bar{f}(v) = \bar{f}(w)$ or $\bar{f}(v)$ and $\bar{f}(w)$ are joined by an edge in Δ^1_{bd}. Hence all simplices $\tilde{\tau}$ in $\widetilde{\Delta}_{\mathrm{in}}$ satisfy the condition (II). We easily see that Z consists of nine rational curves meeting as in Figure 16, that $X \setminus Z$ consisits of three connected compnents each of which is biholomorphic to the product of a cycle of two rational curves and the punctured disc $\{z \in \mathbf{C} \mid 0 < |z| < 1\}$ and that there exist holomorphic functions on U which give the projections to the second factors of the products. Hence any fiber of the restriction to $V \cap (X \setminus Z)$ of the holomorphic map π of the above theorem, is a cycle of two rational curves.

7 Cusp singularities with irreducible exceptional sets

In this section, restricting ourselves to the case $r = 3$, $\Delta^1_{\mathrm{bd}} = \emptyset$, $\Delta^0 = \Delta^0_{\mathrm{f}}$ and $|\Delta^0| = 1$, we prove the following:

Theorem 25. *For any negative even number (resp. negative integer) e, there exists a 3-dimensional cusp singularity with a resolution satisfying the following conditions (1) and (2).*

(1) The exceptional set is irreducible.

(2) The dual graph of the exceptional set is a triangulation of an orientable (resp. a non-orientable) compact topological surface T with $\chi(T) = e$.

We show the existence of a triangulation Δ with one vertex of a compact topological surface with an automorphism of order 2 or 3, and a $\mathbf{Z}$-weight on Δ which satisfies the condition (M-i) and is symmetric around only one vertex. The numbers of triangles and edges of such a triangulation are $2-2e$ and $3-3e$, respectively, where $e = \chi(\Delta)$.

Lemma 26. *For any multiple l of 6 not smaller than 12, there exists a sequence of l integers a_1, a_2, $\ldots$, a_l satisfying the condition (M-i) and the following two conditions.*
(1) $a_i \leq -1$ and if $a_i = -1$, then $a_{i\pm1} < -1$ for $1 \leq i \leq l$, where $a_0 = a_l$, $a_{l+1} = a_1$.
(2) If $i \equiv j (\mathrm{mod}\ l/6)$, then $a_i = a_j$.

Proof. For $l = 12$, the sequence -1, -3, -1, -3, $\ldots$ satisfies the conditions. For $l = 18$, repeat 6 times the inverse operation of (*) in Section 1 on the above sequence so that the condition (2) still holds, as -2, -1, -4, -2, -1, -4, $\ldots$. For $l \geq 24$, we obtain a desired sequence in a similar manner. $\square$

Lemma 27. *For any negative even number e, there exist a triangulation Δ with one vertex of an orientable compact topological surface T with $\chi(T) = e$, and an automorphism of Δ of order 2 preserving the orientation of T.*

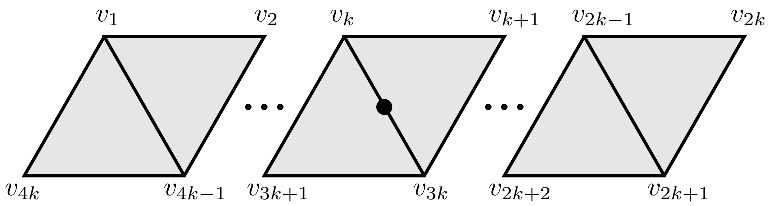


Figure 17

Proof. Let $k = 1 - \frac{e}{2}$ and let Δ be the triangulation obtained by gluing the edges $\overline{v_i v_{i+1}}$ and $\overline{v_{i+2k} v_{i+1+2k}}$ of the development in Figure 17 together as follows, for $1 \leq i \leq 2k$, where $v_{4k+1} = v_1$. Identifying the points $tv_i + (1-t)v_{i+1}$ and $tv_{i+1+2k} + (1-t)v_{i+2k}$ for $0 \leq t \leq 1$. Then Δ has only one vertex, because $\gcd(4k, 2k \pm 1) = 1$. Moreover, Δ is a triangulation of an orientable compact topological surface T with $\chi(T) = e$, because Δ has $4k - 2 = 2 - 2e$ triangles. In the development in Figure 17, the rotation sending v_i to v_{2k+i} gives an automorphism of Δ of order 2 preserving orientation. $\square$

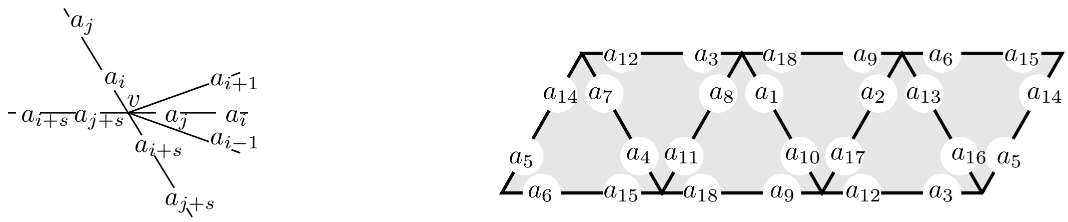


Figure 18

Let Δ and g be a triangulation and an automorphism in the above lemma. Let v be the vertex of Δ and let V be a small neighborhood of v. Since $gv = v$, we may assume that $gV = V$. Let $s = 3 - 3e$. We may assume that the intersection of V with each edge of Δ consists of two connected components. Then the number of pieces of edges in V is $2s$. Let e_1, e_2, $\ldots$, e_{2s} be the pieces and assume that e_j are adjacent to e_{j+1}. If $ge_1 = e_j$, then $ge_2 = e_{j+1}$, $ge_3 = e_{j+2}$, $\ldots$ and $ge_j = e_{2j-1}$. Since $g^2 = 1$, $j = s+1$. Hence $ge_i = e_{s+i}$. Obviously, there exists a simplicial system which induces Δ and $\Delta^0 = \Delta^0_{\mathrm{in}} = \{v\}$. Let τ_i, β_i, β'_i ($1 \leq i \leq 2s$) be as in Lemma 1 for $\bar{\tau} = v$ and let w_i be the vertices of β_i with $w_i \neq \tau_i$. Let a_1, a_2, $\ldots$, a_{2s} be a sequence of integers in Lemma 26. We define $\phi(\beta_i, w_i) = a_i$ (see Figure 18, the right is for $e = -2$). Note that if a_i and a_j are attached to the both sides of an edge e, then a_{i+s} and a_{j+s} are attached to those of ge. Since $a_i = a_{i+s}$ for $1 \leq i \leq s$ and by Proposition 2, ϕ satisfies the condition

(M-i) and symmetric. Moreover, the assumption of Proposition 21 is satisfied, by the condition (1) of Lemma 26.

Lemma 28. *For any negative integer e, there exist a triangulation Δ with one vertex of a non-orientable compact topological surface T with $\chi(T) = e$, and an automorphism of Δ of order* 3.

Proof. For $e = -1$, we can obtain a desired triangulation from the development in Figure 19, glueing two edges joined by a thin curved line together, so that the orientation changes if a small circle is attached on the line. Also for $e = -4$, that in Figure 21 gives a desired one, where Figure 21 is obatined from Figure 19 inserting pieces of Figure 20 at the three arrowed edges. For $e = -7, -10, \ldots$, we can obtain desired ones in a similar manner.

For $e = -2$ and -3, the developments in Figure 22 and 23, respectively, give desired ones. Also for $e = -5, -6, -8, -9, \ldots$, we can obtain desired ones as above. $\square$

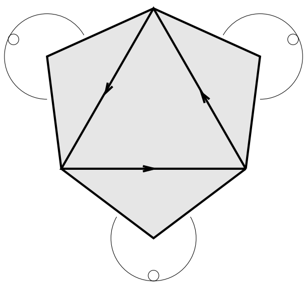


Figure 19

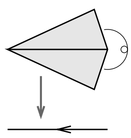


Figure 20

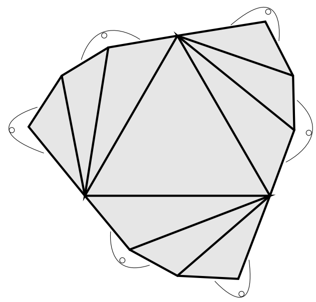


Figure 21

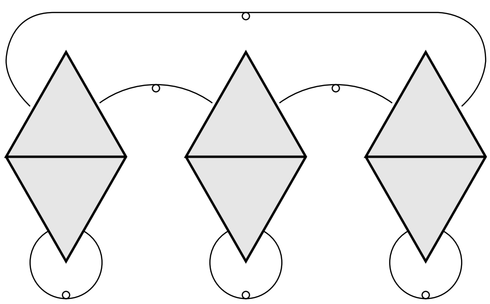


Figure 22

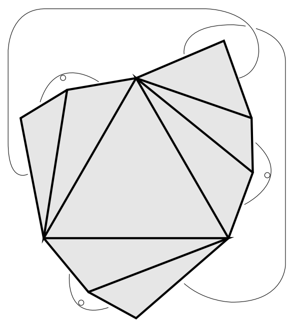


Figure 23

Let Δ and g be a triangulation and an automorphism, respectively in the above lemma. Let v be the vertex of Δ and let V be a small neighborhood of v. Since $gv = v$, we may assume that $gV = V$. Let $s = 2 - 2e$. We may assume that the intersection of V with each edge of Δ consists of two connected components. Then the number of pieces of edges in V is $3s$. Let $e_1, e_2, \ldots, e_{3s}$ be the pieces and assume that e_j are adjacent to e_{j+1}. Then $ge_i = e_{i+s}$ or e_{i+2s}. Let $a_1, a_2, \ldots, a_{3s}$ be a sequence of integers in Lemma 26. We use them as the value of $\mathbf{Z}$-weight ϕ of Δ (see Figure 24, the right is for $e = -1$). Since $a_i = a_{i+s} = a_{i+2s}$ for $1 \leq i \leq s$ and by Proposition 2, ϕ satisfies the condition (M-i). Moreover, ϕ is symmetric, because $a_i = a_{i+3s/2}$ for $1 \leq i \leq 3s/2$.

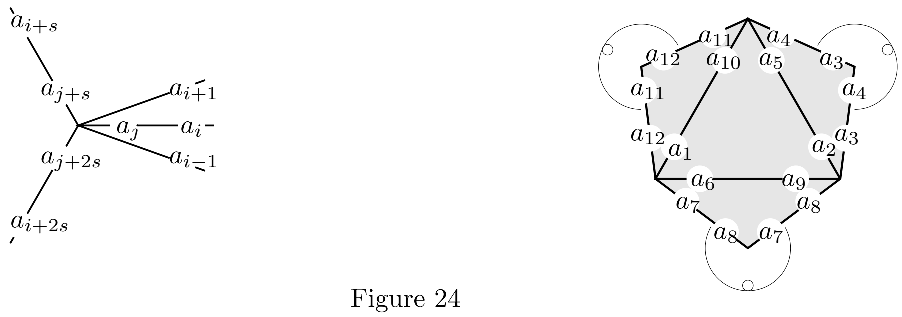


Figure 24

8 An example of a compact complex manifold V with $\pi_1(V) \simeq \mathbf{Z}$

In this section, we give an example of a simplicial system $(A, B, \{l_b\}_{b\in B})$ and a $\mathbf{Z}$-weight such that the induced topological space Δ is a simplicial decomposition with one vertex of a compact topological manifold T with $\pi_1(T) \simeq \mathbf{Z}$ and that $\Delta^{r-3} = \Delta_{\mathrm{f}}^{r-3}$. To avoid complications, we use same symbols for vertices on different simplices which are glued together by l_b for some b in B.

Let $\alpha_i = \overline{v_0 v_1 \cdots v_{i-1} v_{i+1} \cdots v_r}$ for $1 \le i \le r$ and let β_i^j be the $(r-2)$-dimensional faces of α_i which do not contain v_j. Let $A^{r-1} = \{\alpha_i \mid 2 \le i \le r\}$ and let

$$\begin{aligned} B &= \{(\beta_i^j, \beta_j^i) \mid 2 \le i \le r, 2 \le j \le r, i \ne j\} \\ &\quad \bigcup \{(\beta_i^0, \beta_{i+1}^1), (\beta_{i+1}^1, \beta_i^0) \mid 2 \le i \le r-1\} \bigcup \{(\beta_r^0, \beta_2^1), (\beta_2^1, \beta_r^0)\}. \end{aligned}$$

Let $l_{(\beta_i^j, \beta_j^i)}(v_k) = v_k$ $(k \ne i, j)$, $l_{(\beta_i^0, \beta_{i+1}^1)}(v_k) = v_{k+1}$ $(k \ne 0, i, r)$, $l_{(\beta_i^0, \beta_{i+1}^1)}(v_r) = v_0$, $l_{(\beta_r^0, \beta_2^1)}(v_k) = v_{k+2}$ $(k \ne 0, r-1, r)$, $l_{(\beta_r^0, \beta_2^1)}(v_{r-1}) = v_0$ and define the other l_b so that the condition (v) holds. Then $(A, B, \{l_b\}_{b\in B})$ is a simplicial system with $\Delta_{\mathrm{bd}}^{r-2} = \emptyset$ and $|\Delta^0| = 1$. Let $\phi(\beta_i^j, v_k) = 1$ $(k \ne 0)$, $\phi(\beta_i^j, v_0) = -1$ for $j \ge 2$, $\phi(\beta_i^0, v_k) = 0$ $(k \ne r)$, $\phi(\beta_i^0, v_r) = -2$ for $i < r$, $\phi(\beta_r^0, v_k) = -1$ $(k \ne r-1)$, $\phi(\beta_r^0, v_{r-1}) = -3$ and define the other values of ϕ so that the condition (vi) holds. Then ϕ satisfies (M-i) around all slices in $\Delta^{r-3} = \Delta_{\mathrm{in}}^{r-3}$ (see Figure 25 $\sim$ 28, where the meaning of the integers attached to edges is as in Figure 29). In the definition of Path(Δ), $(\beta_i^j, \beta_j^i)(\beta_j^k, \beta_k^j) \sim (\beta_i^k, \beta_k^i)$ for integers i, j, k greater than 1 and different to each other (see Figure 25), and $(\beta_i^0, \beta_{i+1}^1) \sim (\beta_i^3, \beta_3^i)(\beta_3^0, \beta_4^1)(\beta_4^{i+1}, \beta_{i+1}^4)$ for $i = 2, 4, \ldots, r-1$ (see Figure 26). Moreover,

$$(\beta_r^0, \beta_2^1) \sim (\beta_r^2, \beta_2^r)(\beta_2^0, \beta_3^1)(\beta_3^0, \beta_4^1)(\beta_4^2, \beta_2^4) \sim (\beta_r^2, \beta_2^r)(\beta_2^3, \beta_3^2)(\beta_3^0, \beta_4^1)(\beta_4^3, \beta_3^4)(\beta_3^0, \beta_4^1)(\beta_4^2, \beta_2^4)$$

(see Figure 27). Choose α_3 as α_0 in the definition of Path(Δ). Then we see by the above three relations that any element in Path(Δ) is equal to

$$[\left((\beta_3^0, \beta_4^1)(\beta_4^3, \beta_3^4)\right)^i], \ [\left((\beta_3^0, \beta_4^1)(\beta_4^3, \beta_3^4)\right)^i (\beta_3^j, \beta_j^3)], \ [\left((\beta_3^4, \beta_4^3)(\beta_4^1, \beta_3^0)\right)^i] \ \text{ or } \ [\left((\beta_3^4, \beta_4^3)(\beta_4^1, \beta_3^0)\right)^i (\beta_3^j, \beta_j^3)]$$

for a non-negative integer i and an integer $j \ne 3$. For examples,

$$\begin{aligned} (\beta_3^5, \beta_5^3)(\beta_5^4, \beta_4^5)(\beta_4^0, \beta_5^1)(\beta_5^2, \beta_2^5) &\sim (\beta_3^4, \beta_4^3)(\beta_4^3, \beta_3^4)(\beta_3^0, \beta_4^1)(\beta_4^5, \beta_5^4)(\beta_5^2, \beta_2^5) \\ &\sim (\beta_3^0, \beta_4^1)(\beta_4^2, \beta_2^4) \sim (\beta_3^0, \beta_4^1)(\beta_4^3, \beta_3^4)(\beta_3^2, \beta_2^3) \end{aligned}$$

$2 \le i < j < k \le r$

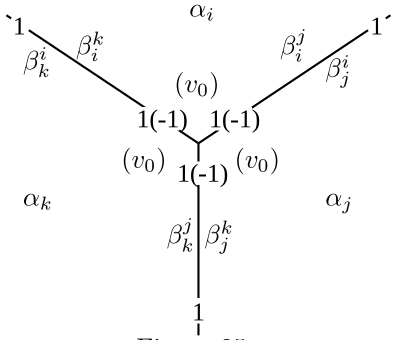


Figure 25

$2 \le i < j < r$

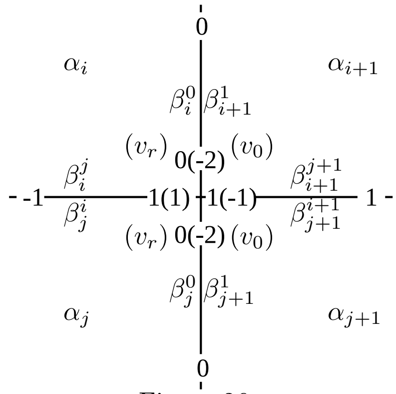


Figure 26

$3 \le i \le r-1$

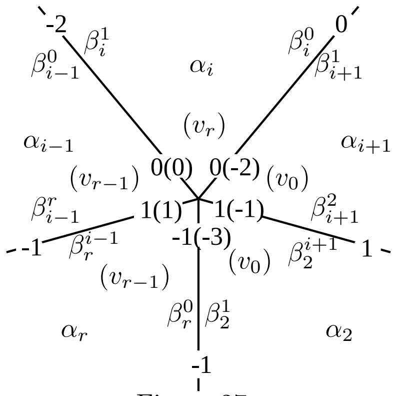


Figure 27

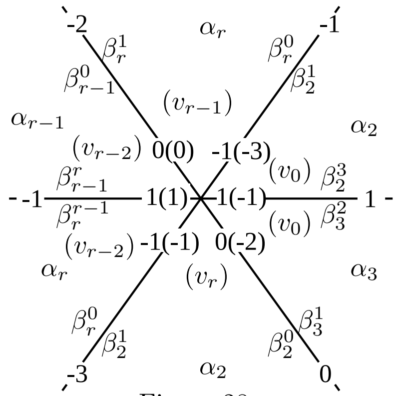
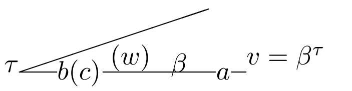


Figure 28



$a = \phi(\beta, v)$
$b = \phi(\beta, u)\ (u \neq v, w)$
$c = \phi(\beta, w)$

Figure 29

In the following, we show that Δ is a simplicial decompostion of a compact topological manifold T with $\pi_1(T) \simeq \mathbf{Z}$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r\}$ be a basis of N and let $\mathbf{e}_0 = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_r$. Let $\bar{H}(x) = \mathrm{pr}(t_0\mathbf{e}_0 + t_1\mathbf{e}_1 + \cdots + t_r\mathbf{e}_r)$ for points $x = t_0v_0 + t_1v_1 + \cdots + t_rv_r$ $(t_i = 0)$ on α_i, where $\mathrm{pr} : N_{\mathbf{R}} \setminus \{0\} \to S^{r-1}$ is the canonical projection. Then $\bar{H}(x) = \bar{H}(l_{(\beta_i^j, \beta_j^i)}(x))$ for points x on β_i^j and $\bar{H}(\alpha_i) \cap \bar{H}(\alpha_j) = \bar{H}(\beta_i^j)$ if $2 \le i < j \le r$. Let g be the element in $\mathrm{GL}(N)$ sending $\mathbf{e}_1, \ldots, \mathbf{e}_{r-1}$ and $\mathbf{e}_r$ to $\mathbf{e}_2, \ldots, \mathbf{e}_r$ and $\mathbf{e}_0$, respectively. Let $E = \mathrm{pr}(\overline{\mathbf{e}_1\mathbf{e}_2 \cdots \mathbf{e}_r})$ and let $F = \bigcup_{i=2}^r \bar{H}(\alpha_i)$. Then $F = \overline{E \setminus gE}$. Hence $F \cup gF \cup \cdots \cup g^iF = \overline{E \setminus g^{i+1}E}$ for $i > 0$. Since $F \cap g^iF = \emptyset$ for $i \ge r+1$, $g^{\mathbf{Z}}$ acts on $D := \bigcup_{i \in \mathbf{Z}} g^iF$ properly discontinuously. Moreover, $D/g^{\mathbf{Z}}$ is a topological manifold, because $g^{\mathbf{Z}}$ acts on D freely. Since $E \setminus g^iE$ is simply connected for $i \ge r+1$, so is D. On the other hand,

$$\bar{H}(v_i) + \bar{H}(v_j) + \sum_{k \in \{0,1,\ldots,r\} \setminus \{i,j\}} \phi(\beta_i^j, v_k)\bar{H}(v_k) = 0$$

for $2 \le i < j \le r$, because $\bar{H}(v_i) = \mathbf{e}_i$ for $0 \le i \le r$. Hence if we define the map h in Section 2 so that $h(w_i) = \mathbf{e}_i$ for the vertices w_i of $\alpha([])$ with $f(w_i) = v_i$ $(i \neq 3)$, then $h(w) = \bar{H}(f(w))$ for the vertices w

on $\alpha([(\beta_3^i,\beta_i^3)])$ $(i=2,4,\dots,r)$. Since $g^{-1}\bar{H}(l_{(\beta_i^0,\beta_{i+1}^1)}(v_j))=\bar{H}(v_j)$ for $1\le j\le r$ and

$$\bar{H}(v_0)+g^{-1}\bar{H}(v_1)+\sum_{k\in\{1,\dots,i-1,i+1,\dots,r\}}\phi(\beta_i^0,v_k)\bar{H}(v_k)=0$$

for $2\le i\le r-1$, $h(w)=g^{-1}\bar{H}(f(w))$ for the vertices w on $\alpha([(\beta_3^i,\beta_i^3)(\beta_i^0,\beta_{i+1}^1)])$. Hence $\bar{h}(\alpha([(\beta_3^0,\beta_4^1)(\beta_4^3,\beta_3^4)]))=g^{-1}(\bar{H}(\alpha_3))$, because $[(\beta_3^2,\beta_2^3)(\beta_2^0,\beta_3^1)]=[(\beta_3^0,\beta_4^1)(\beta_4^3,\beta_3^4)]$. Moreover,

$$\bar{h}(\alpha([((\beta_3^0,\beta_4^1)(\beta_4^3,\beta_3^4))^i(\beta_3^j,\beta_j^3)]))=g^{-i}(\bar{H}(\alpha_j))\text{ and }\bar{h}(\alpha([((\beta_3^4,\beta_4^3)(\beta_4^1,\beta_3^0))^i(\beta_3^j,\beta_j^3)]))=g^{i}(\bar{H}(\alpha_j))$$

for any positive integer i. Therefore, the image of $\bar{h}:\widetilde{\Delta}\to S^{r-1}$ is equal to D. Moreover, $\bar{h}$ is injective, because D is a simply connected topological manifold and the restriction of $\bar{h}$ to the complement of the $(r-4)$-dimensional slices of $\widetilde{\Delta}$, is locally homeomorphic, by Proposition 10. Hence Δ is homeomorphic to $D/g^{\mathbf{Z}}$ and $\mathrm{Path}^0(\Delta)=\{[((\beta_3^0,\beta_4^1)(\beta_4^3,\beta_3^4))^i]\mid i\in\mathbf{Z}\}\simeq\mathbf{Z}$.

Next, let

$$R=\{t_1\mathbf{e}_1+t_2\mathbf{e}_2+\cdots+t_r\mathbf{e}_r\mid t_i\ge 0, t_1+t_2+\cdots+t_r\ge 1\}.$$

Then $g^rR\subset\mathrm{Int}(R)$. Let $S=\overline{R\setminus g^rR}$. Then $S\cap g^rS=\partial(g^rR)$ and $S\cap g^{ir}S=\emptyset$ for $i\ge 2$. Hence $g^{\mathbf{Z}}$ acts on $\bigcup_{i\in\mathbf{Z}}g^{ir}S$ properly discontinuously. Let Y be the toric variety associated to the fan $\Sigma(\Delta)$. Let $\widetilde{X}$ and $\mathrm{ord}:T_N\to N_{\mathbf{R}}$ be as in Section 4. $\widetilde{V}:=\bigcup_{i\in\mathbf{Z}}\overline{\mathrm{ord}^{-1}(g^{ir}S)}$ is an open set of Y containing $\widetilde{X}$ on which $g^{\mathbf{Z}}$ acts properly discontinuously and freely. Hence $V=\widetilde{V}/g^{\mathbf{Z}}$ is a complex manifold containing the irreducible analytic set $X:=\widetilde{X}/g^{\mathbf{Z}}$. Since $\overline{\mathrm{ord}^{-1}(S)}$ is compact, so is V. Since $S^i:=\bigcup_{j=0}^{i}g^{jr}S$ is simply connected for any positive integer i, $\pi_1(\mathrm{ord}^{-1}(S^i))\simeq N$. We see that $\pi_1(\overline{\mathrm{ord}^{-1}(S^i)})\simeq\{1\}$ in a similar manner as the proof of [4, Proposition 1.9]. Hence $\widetilde{V}$ is simply connected.

9 Deformations of group actions

Let Σ and Γ be a fan in N and a subgroup of $GL(N)$, respectively, satisfying the condition (F) in Introduction. T_N acts on $Y:=T_N\mathrm{emb}(\Sigma)$ by multiplication and $\gamma\circ t=\gamma t\circ\gamma$ for t in T_N and γ in $GL(N)$, because $\gamma(ty)=(\gamma t)(\gamma y)$ for y in Y. Let

$$Z^1(\Gamma,T_N)=\{t:\Gamma\to T_N\mid t(\gamma\delta)=t(\gamma)(\gamma t(\delta))\text{ for }\gamma,\delta\in\Gamma\}\supset B^1(\Gamma,T_N)=\{\partial t_0\mid t_0\in T_N\},$$

where $(\partial t_0)(\gamma)=(\gamma t_0)t_0^{-1}$, and let $H^1(\Gamma,T_N)=Z^1(\Gamma,T_N)/B^1(\Gamma,T_N)$. For any element t in $Z^1(\Gamma,T_N)$, $\Gamma(t):=\{t(\gamma)\circ\gamma\mid\gamma\in\Gamma\}$ is a group acting on Y and isomorphic to Γ. If an open set $\widetilde{U}$ of Y is Γ-invariant, then $t_0^{-1}\widetilde{U}$ is $\Gamma(\partial t_0)$-invariant and $\widetilde{U}/\Gamma\simeq t_0^{-1}\widetilde{U}/\Gamma(\partial t_0)$ for any element t_0 in T_N. For an element t in $Z^1(\Gamma,T_N)$, let $\hat{t}$ be the element in $Z^1(\Gamma,N_{\mathbf{R}}):=\{u:\Gamma\to N_{\mathbf{R}}\mid u(\gamma\delta)=u(\gamma)+\gamma u(\delta)\text{ for }\gamma,\delta\in\Gamma\}$ defined by $\hat{t}(\gamma)=(\mathrm{ord}\circ t)(\gamma)$. Then $\Gamma(\hat{t}):=\{\hat{t}(\gamma)\circ\gamma\mid\gamma\in\Gamma\}$ acts on $\mathrm{Mc(N,\Sigma)}=Y/CT_N$, where $\hat{t}(\gamma)v=v+\hat{t}(\gamma)$ for v in $N_{\mathbf{R}}$. If an open set O of $\mathrm{Mc(N,\Sigma)}$ is $\Gamma(\hat{t})$-invariant and $\Gamma(\hat{t})$ acts on O properly discontinuously, then so does $\Gamma(t)$ on $\mathrm{ord}^{-1}(O)$. In particular, $\Gamma(t)$ acts properly discontinuously on $\widetilde{U}$ in Section 6, if the image $t(\Gamma)$ of t is contained in CT_N.

Proposition 29. *Theorem 23 is valid also for $U(t):=\widetilde{U}/\Gamma(t)$ and $Z(t):=\widetilde{Z}/\Gamma(t)$ instead of U and Z, respectively, if the image $t(\Gamma)$ of an element t in $Z^1(\Gamma,T_N)$ is contained in CT_N.*

Proof. For every element x in M and every element γ in Γ, $|t(\gamma)^*\mathbf{e}^x|=|\,(t(\gamma)(x))\,\mathbf{e}^x|=|\mathbf{e}^x|$. Hence $\sum_{x\in x_0\Gamma}|\mathbf{e}^{2x}|$ is $\Gamma(t)$-invariant for every element x_0 in M. Therefore, the proof of Theorem 23 is valid also for $U(t)$ and $Z(t)$. $\square$

Proposition 30. *Theorem 24 is valid also for $U(t)$, $X(t):=(\widetilde{X}\cap\widetilde{U})/\Gamma(t)$ and $Z(t)$ instead of U, X and Z, respectively, if the image $t(\Gamma)$ of an element t in $Z^1(\Gamma,T_N)$ is contained in $T_N(k):=\{t\in$*

$T_N \mid t^k = 1\}$ *for a positive integer* k.

Proof. For every element x in M and every element γ in Γ, $t(\gamma)(kx) = (t(\gamma)(x))^k = 1$. Hence $\sum_{x \in kx_0\Gamma} \mathbf{e}^x$ is $\Gamma(t)$-invariant for every element x_0 in M. Therefore, the proof of Theorem 24 is valid also for $U(t)$, $X(t)$ and $Z(t)$, replacing $\tilde{f}^{x_0}$ and $\tilde{f}^{x_\mu}$ with $\tilde{f}^{kx_0}$ and $\tilde{f}^{kx_\mu}$, respectively. $\square$

Let Γ be an infinte cyclic group generated by an element γ in $GL(N)$. We easily see that $H^1(\Gamma, T_N) = 1$, if and only if $\det(\gamma - I_r) \neq 0$. This condition is satisfied by the group in Section 8 and those which give 2-dimensional Hilbert modular cusp singularities. Higher-dimensional Hilbert modular cusp singularities are obtained from free abelian groups generated by elements in $GL(N)$ which do not have eigenvalues equal to 1.

Proposition 31. *Let* Γ *be a free abelian group generated by elements* γ_1, γ_2, …, γ_l *in* $GL(N)$ *with* $\det(\gamma_i - I_r) \neq 0$. *Then* $H^1(\Gamma, T_N)$ *is a finite group. Moreover,* $H^1(\Gamma, T_N) = 1$, *if* $\det(\gamma_1 - I_r) = \pm 1$.

Proof. Let g_i be the endomorphism of T_N sending t_0 to $(\gamma_i t_0)t_0^{-1}$ for $1 \leq i \leq l$. Then g_i is surjective and the degree is equal to $|\det(\gamma_i - I_r)|$. Let t be any element in $Z^1(\Gamma, T_N)$. Then $g_i t(\gamma_j) = g_j t(\gamma_i)$, because $t(\gamma_i\gamma_j) = t(\gamma_j\gamma_i)$. Let t_0 be an element in T_N with $g_1 t_0 = t(\gamma_1)^{-1}$ and let $t' = t\partial t_0$. Then $t'(\gamma_1) = t(\gamma_1)g_1 t_0 = 1$. Hence $g_1 t'(\gamma_i) = g_i t'(\gamma_1) = 1$ for $2 \leq i \leq l$. Therefore, the assertion holds. $\square$

We give another example of Γ which is a free abelian group. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r\}$ be a basis of N and let a_1, a_2, …, a_{r-1} be positive integers. Let Γ be the group generated by the elements γ_1, γ_2, …, γ_{r-1} in $GL(N)$ defined as follows: $\gamma_i \mathbf{e}_j = \mathbf{e}_j$ for $1 \leq j \leq r-1$, $\gamma_i \mathbf{e}_r = \mathbf{e}_r + a_i \mathbf{e}_i$. Let v be an element in $Z^1(\Gamma, N_{\mathbf{R}})$. Then $(\gamma_i - I_r)v(\gamma_j) = (\gamma_j - I_r)v(\gamma_i)$, because $\gamma_i\gamma_j = \gamma_j\gamma_i$. Let $v(\gamma_j) = \sum_{k=1}^{r} c_{jk}\mathbf{e}_k$. Then $(\gamma_i - I_r)v(\gamma_j) = c_{jr}a_i\mathbf{e}_i$. Hence $c_{jr} = 0$ if $r \geq 3$. On the other hand, if v is in $B^1(\Gamma, N_{\mathbf{R}})$, then $v(\gamma_i) = ba_i\mathbf{e}_i$ for a real number b. First, we consider the case that $r = 2$. Let $\{\mathbf{e}_1^*, \mathbf{e}_2^*\}$ be the basis of M dual to $\{\mathbf{e}_1, \mathbf{e}_2\}$. Let λ be a non-zero complex number and let t be the element in $Z^1(\Gamma, T_N)$ with $t(\gamma_1)(\mathbf{e}_1^*) = 1$, $t(\gamma_1)(\mathbf{e}_2^*) = \lambda$. Assume that $|\lambda| = 1$. Then $\Gamma(\hat{t}) = \Gamma$, because $\hat{t}(\gamma_1) = (-\log|\lambda|)\mathbf{e}_2 = 0$. Let

$$\Sigma = \{\{0\}, \mathbf{R}_{\geq 0}(i\mathbf{e}_1 + \mathbf{e}_2), \mathbf{R}_{\geq 0}((i-1)\mathbf{e}_1 + \mathbf{e}_2) + \mathbf{R}_{\geq 0}(i\mathbf{e}_1 + \mathbf{e}_2) \mid i \in \mathbf{Z}\}$$

and let $O = |\Sigma| \setminus \{0\} = \{y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \in N_{\mathbf{R}} \mid y_2 > 0\}$. Then $\Gamma(t)$ acts on the open set $\widetilde{U} := \mathrm{Int}(\overline{\mathrm{ord}^{-1}(O)})$ of Y properly discontinuously. If λ is a primitive k-th root of the unity, then $U(t) := \widetilde{U}/\Gamma(t)$ is an elliptic surface over an open disk with only one singular fiber of type kI_{a_1}, because $\mathbf{e}^{k\mathbf{e}_2^*}$ is $\Gamma(t)$-invariant and gives a fibration on $U(t)$.

Lemma 32. *If* $|\lambda| = 1$ *and* $\lambda^k \neq 1$ *for any positive integer* k, *then the above* $U(t)$ *contains no compact analytic curves except those contained in* $X(t) := (Y \setminus T_N)/\Gamma(t)$.

Proof. Suppose that $U(t)$ contains a compact analytic curve E not contained in $X(t)$. Let E' be a connected component of $[\widetilde{U} \to U(t)]^{-1}(E)$. Then $|\mathbf{e}^{\mathbf{e}_2^*}|$ is constant on E', because it is a $\Gamma(t)$-invariant plurisubharmonic function on $\widetilde{U}$. Hence so is $\mathbf{e}^{\mathbf{e}_2^*}$. Since $\mathbf{e}^{\mathbf{e}_2^*}$ does not vanish on T_N and $t(\gamma_1^k)(\mathbf{e}_2^*) = (t(\gamma_1)(\mathbf{e}_2^*))\,(\gamma_1 t(\gamma_1^{k-1}))(\mathbf{e}_2^*) = \lambda t(\gamma_1^{k-1})(\mathbf{e}_2^*\gamma_1) = \lambda t(\gamma_1^{k-1})(\mathbf{e}_2^*) = \cdots = \lambda^k \neq 1$ for any positive integer k, $\{\gamma \in \Gamma(t) \mid \gamma E' = E'\} = \{1\}$. It contradicts the assumption that E is compact. $\square$

Assume that $|\lambda| < 1$. Let

$$\Sigma = \{\{0\}, \mathbf{R}_{\geq 0}(i\mathbf{e}_1 + \mathbf{e}_2), \mathbf{R}_{\geq 0}((i-1)\mathbf{e}_1 + \mathbf{e}_2) + \mathbf{R}_{\geq 0}(i\mathbf{e}_1 + \mathbf{e}_2), \mathbf{R}_{\geq 0}\mathbf{e}_1 \mid i \in \mathbf{Z}\}.$$

Then the closure of

$$\{y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \in N_{\mathbf{R}} \mid -a_1y_2 \leq y_1 \leq 0\} \cup \{y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \in N_{\mathbf{R}} \mid 0 \leq y_1, 0 \leq y_2 \leq b\}$$

in $\mathrm{Mc}(\mathrm{N}, \Sigma)$ is a fundamental domain of the action of $\Gamma(\hat{t})$ on $\mathrm{Mc}(\mathrm{N}, \Sigma)$, because $\hat{t}(\gamma_1) = b\mathbf{e}_2$, where $b = -\log|\lambda|$. Hence $\Gamma(\hat{t})$ acts on $\mathrm{Mc}(\mathrm{N}, \Sigma)$ properly discontinuously. $Y/\Gamma(t)$ is a parabolic Inoue surface (see [4, 4.3.(2)(b)]). Next, we consider the case that $r \geq 3$.

Proposition 33. *Let Γ be as above. For any element v in $Z^1(\Gamma, N_{\mathbf{R}})$, there exists a real number s_0 such that $\Gamma(v)$ acts on $C := \{y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + \cdots + y_r\mathbf{e}_r \in N_{\mathbf{R}} \mid y_r > s_0\}$ properly discontinuously.*

Proof. For any real number s, $v(\gamma_i) \circ \gamma_i$ sends $s\mathbf{e}_r$ to $s\mathbf{e}_r + a_i s\mathbf{e}_i + v(\gamma_i)$. We can express $v(\gamma_i) = \sum_{k=1}^{r-1} c_{ik}\mathbf{e}_k$. Let $d(s)$ be the determinant of the $(r-1) \times (r-1)$ matrix whose (i,j) entry is c_{ij} (resp. $a_i s + c_{ii}$) for $i \neq j$ (resp. $i = j$). If $d(s) \neq 0$, then $\Gamma(v)$ acts properly discontinuously on the hyperplane in $N_{\mathbf{R}}$ defined by $y_r = s$. On the other hand, $d(s)$ is a polynomial of s whose coefficient of s^{r-1} is $a_1 a_2 \cdots a_{r-1} > 0$. Hence there exists a real number s_0 such that $d(s) > 0$ for $s > s_0$. $\square$

Let $\Sigma = \{\text{faces of } \gamma\sigma(i) \mid \gamma \in \Gamma', i \in I\}$, where

$$\sigma(i_1, i_2, \ldots, i_{r-1}) = \mathbf{R}_{\geq 0}\mathbf{e}_r + \mathbf{R}_{\geq 0}(\mathbf{e}_r + \mathbf{e}_{i_1}) + \cdots + \mathbf{R}_{\geq 0}(\mathbf{e}_r + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_{r-1}}),$$

I is the set of permutations of $\{1, 2, \ldots, r-1\}$ and Γ' is the group Γ for the case $a_1 = a_2 = \cdots = a_{r-1} = 1$. Then Σ is a Γ-invariant fan and $|\Sigma| = \{y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + \cdots + y_r\mathbf{e}_r \in N_{\mathbf{R}} \mid y_r > 0\} \cup \{0\}$. Let t be any element in $Z^1(\Gamma, T_N)$. Then $\Gamma(t)$ acts on $\widetilde{U} := \mathrm{Int}(\overline{\mathrm{ord}^{-1}(C)})$ properly discontinuously, where C is as in the above proposition for $v = \hat{t}$. Since $\mathbf{e}_k^*\gamma_k = \mathbf{e}_k^* + a_k\mathbf{e}_r^*$ and $\mathbf{e}_k^*\gamma_j = \mathbf{e}_k^*$ if $j \neq k$, $\left(\gamma_i t(\gamma_j) t(\gamma_j)^{-1}\right)(\mathbf{e}_k^*) = t(\gamma_j)(\mathbf{e}_k^*\gamma_i)\left(t(\gamma_j)(\mathbf{e}_k^*)\right)^{-1} = t(\gamma_j)(a_i\mathbf{e}_r^*)$ or 1, accordingly as $i = k$ or $i \neq k$. Hence $t(\gamma_j)(a_i\mathbf{e}_r^*) = 1$ for $i \neq j$, because $\gamma_i t(\gamma_j) t(\gamma_j)^{-1} = \gamma_j t(\gamma_i) t(\gamma_i)^{-1}$. Therefore, there exists a positive integer a_0 such that $\mathbf{e}^{a_0\mathbf{e}_r^*}$ is $\Gamma(t)$-invariant, i.e., $\widetilde{U}/\Gamma(t)$ is a degenerating family of complex tori.

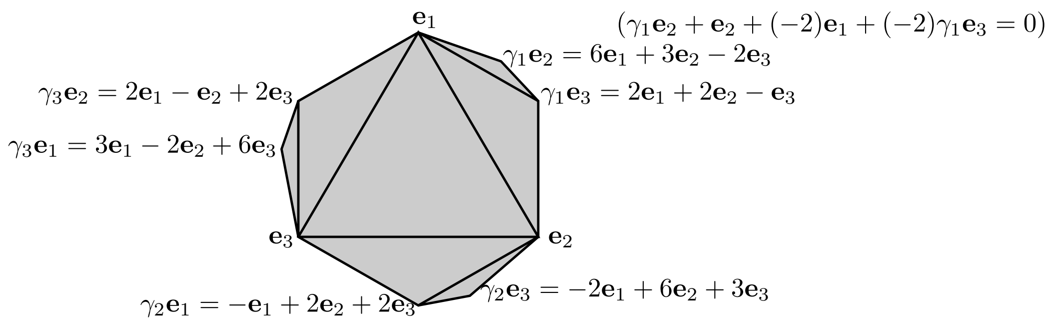


Figure 30

Now, we consider the case that Γ is a free group. Let $\gamma_1, \gamma_2, \ldots, \gamma_l$ be generators of a free group $\Gamma \subset GL(N)$. Then for any elements $t_1, t_2, \ldots, t_l$ in T_N, there exists the element t in $Z^1(\Gamma, T_N)$ with $t(\gamma_i) = t_i$. Let Σ and Γ be a fan and a group, respectively, obtained from Δ in Example 7. Let $\overline{v_1v_2v_3}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be as in Example 7. Then Γ is the free group generated by γ_1 and γ_2 defined as follows: $\gamma_1\mathbf{e}_1 = \mathbf{e}_1$, $\gamma_1\mathbf{e}_2 = 6\mathbf{e}_1 + 3\mathbf{e}_2 - 2\mathbf{e}_3$, $\gamma_1\mathbf{e}_3 = 2\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$, $\gamma_2\mathbf{e}_1 = -\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3$, $\gamma_2\mathbf{e}_2 = \mathbf{e}_2$, $\gamma_2\mathbf{e}_3 = -2\mathbf{e}_1 + 6\mathbf{e}_2 + 3\mathbf{e}_3$ (see Figure 30). Let $\widetilde{U}$, $\widetilde{X}$ and $\widetilde{Z}$ be as in Example 7. Let t be an element in $Z^1(\Gamma, T_N)$ with $t(\gamma_1), t(\gamma_2) \in CT_N$. Let $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$ be the basis of M dual to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, let $x_i = \mathbf{e}_1^* + \mathbf{e}_2^* + \mathbf{e}_3^* - \mathbf{e}_i^*$ and let $\gamma_3 = \gamma_1^{-1}\gamma_2^{-1}$. Then $\langle x_i, \mathbf{e}_i\rangle = 0$ and $x_i\gamma_i = x_i$ for $1 \leq i \leq 3$. Hence we see by Proposition 13 and Lemma 32 that $X_i := (\overline{\mathrm{orb}(\mathbf{R}_{\geq 0}\mathbf{e}_i)} \cap \widetilde{U})/\langle t(\gamma_i) \circ \gamma_i\rangle$ contains compact analytic curves not notained in $E_i := (\overline{\mathrm{orb}(\mathbf{R}_{\geq 0}\mathbf{e}_i)} \setminus \mathrm{orb}(\mathbf{R}_{\geq 0}\mathbf{e}_i))/\langle t(\gamma_i) \circ \gamma_i\rangle$, if and only if $t(\gamma_i)(x_i)$ is a root of the unity. On the other hand, the compact curve E_i is not contractible in X_i, because it is a cycle of two rational curves with the self-intersection number -2. Therefore, $X(t) := (\widetilde{X} \cap \widetilde{U})/\Gamma(t)$ is contractible in $U(t) := \widetilde{U}/\Gamma(t)$ only if all $t(\gamma_i)(x_i)$ are roots of the unity. Conversely, if $t(\gamma_1)^k = t(\gamma_2)^k = 1$ for a positive integer k, then $X(t) \cap V$ is contractible in a neighborhood V of $Z(t) := \widetilde{Z}/\Gamma(t)$, by Proposition 30. Let

k, l, m be positive integers and let t be the element in $Z^1(\Gamma, T_N)$ defined as follows: $t(\gamma_1)(\mathbf{e}_1^*) = \epsilon_k^{-1}\epsilon_m^{-1}$, $t(\gamma_1)(\mathbf{e}_2^*) = \epsilon_k$, $t(\gamma_2)(\mathbf{e}_3^*) = \epsilon_l$, $t(\gamma_1)(\mathbf{e}_3^*) = t(\gamma_2)(\mathbf{e}_1^*) = t(\gamma_2)(\mathbf{e}_2^*) = 1$, where $\epsilon_k = \exp(2\pi\sqrt{-1}/k)$. Then $t(\gamma_1)(x_1) = \epsilon_k$, $t(\gamma_2)(x_2) = \epsilon_l$ and $t(\gamma_3)(x_3) = \epsilon_m$. Hence X_1, X_2 and X_3 are elliptic surfaces with multiple fibers of type kI_2, lI_2 and mI_2, respectively.

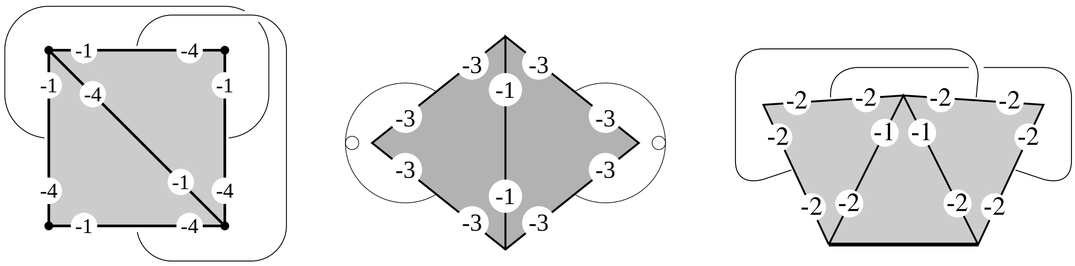


Figure 31

Three Δ in Figure 31 give other examples with free groups Γ.

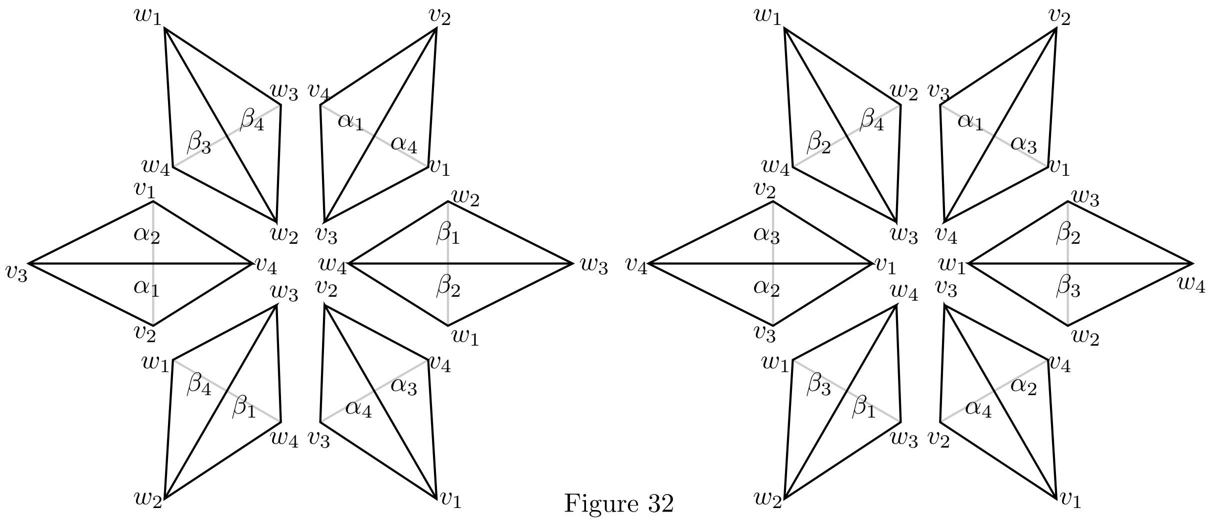


Figure 32

Finally, we give an example such that Γ is not free nor abelian. Let $r = 4$ and let $A^3 = \{\alpha, \beta\}$, where $\alpha = \overline{v_1v_2v_3v_4}$ and $\beta = \overline{w_1w_2w_3w_4}$. Let α_i and β_i be the 2-dimensional faces of α and β which do not contain v_i and w_i, respectively. Let $B = \{(\alpha_i, \beta_{5-i}),\ (\beta_i, \alpha_{5-i}) \mid 1 \le i \le 4\}$. Let $l_{(\alpha_i,\beta_{5-i})}(v_j) = w_{[j-i]}$ and let $l_{(\beta_i,\alpha_{5-i})}(w_j) = v_{[j-i]}$, where $[k] = k$ (resp. $k+5$), if $k > 0$ (resp. $k < 0$). Then $(A, B, \{l_b\}_{b\in B})$ is a simplicial system and the induced topological space Δ has only one vertex whose compliment is a topological manifold, and two 1-dimensional slices at each of which six 3-dimensional slices meet (see Figure 32). Let ϕ be the $\mathbf{Z}$-weight on Δ whose values are all equal to -1. Then the condition (M-i) is satisfied around the two 1-dimensional slices of Δ. Moreover, the conditions in Theorem 12 and Proposition 21 are satisfied. Choose α as α_0 in the definition of Path(Δ) in Section 2. Let $h(u_i) = \mathbf{e}_i$ for $1 \le i \le 4$ in the definition of the map $h : \widetilde{\Delta}^0 \to N$, where u_i are the vertices of a simplex $\overline{u_1u_2u_3u_4}$ of $\widetilde{\Delta}$ with $f(u_i) = v_i$. Let $\gamma_i = \rho([(\alpha_1, \beta_4), (\beta_{[5-i]}, \alpha_i)])$ for $i = 2, 3, 4$. Then

$$\gamma_2 = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ \gamma_3 = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{pmatrix},\ \gamma_4 = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix},$$

$\Gamma = \langle \gamma_2, \gamma_3, \gamma_4 \rangle$ and the relations $\gamma_4\gamma_3^{-1}\gamma_4\gamma_2 = 1$, $\gamma_3\gamma_2^{-1}\gamma_4\gamma_2^{-1}\gamma_3 = 1$ hold, because

$$((\alpha_1, \beta_4), (\beta_1, \alpha_4), (\alpha_3, \beta_2), (\beta_1, \alpha_4), (\alpha_1, \beta_4), (\beta_3, \alpha_2)) \sim (),$$

$$((\alpha_1, \beta_4), (\beta_2, \alpha_3), (\alpha_2, \beta_3), (\beta_1, \alpha_4), (\alpha_2, \beta_3), (\beta_2, \alpha_3)) \sim ()$$

(see Figure 32). In the following calculation, we use $\mathbf{C}^4/\mathbf{Z}^4$ instead of T_N for simplicity. Let v be an element in $Z^1(\Gamma, \mathbf{C}^4/\mathbf{Z}^4)$. Then we obtain the relations: $\gamma_4\gamma_3^{-1}\gamma_4 v(\gamma_2) - \gamma_4\gamma_3^{-1}v(\gamma_3) + (I_4 + \gamma_4\gamma_3^{-1})v(\gamma_4) = 0$ and $-\gamma_3\gamma_2^{-1}(I_4 + \gamma_4\gamma_2^{-1})v(\gamma_2) + (I_4 + \gamma_3\gamma_2^{-1}\gamma_4\gamma_2^{-1})v(\gamma_3) + \gamma_3\gamma_2^{-1}v(\gamma_4) = 0$. Hence if $v(\gamma_4) = 0$, then

$$v(\gamma_3) = \gamma_4 v(\gamma_2), \quad (-(I_4 + \gamma_4\gamma_2^{-1}) + \gamma_2\gamma_3^{-1}(I_4 + \gamma_3\gamma_2^{-1}\gamma_4\gamma_2^{-1})\gamma_4)v(\gamma_2) = 0.$$

Since

$$-I_4 - \gamma_4\gamma_2^{-1} + \gamma_2\gamma_3^{-1}\gamma_4 + \gamma_4\gamma_2^{-1}\gamma_4 = \begin{pmatrix} 0 & -2 & 0 & -4 \\ 0 & 0 & 2 & 4 \\ 0 & 4 & 0 & 8 \\ -2 & 2 & -2 & 2 \end{pmatrix},$$

$v(\gamma_2)$ is in $K := \left(\mathbf{C}^t(1, -2, -2, 1) + \frac{1}{2}\mathbf{Z}^4\right)/\mathbf{Z}^4$, if $v(\gamma_4) = 0$. Hence $L := \{v \in Z^1(\Gamma, \mathbf{C}^4/\mathbf{Z}^4) \mid v(\gamma_4) = 0\} \simeq K$. Since $|\gamma_4 - I_4| = 3$, there exists an element v in $B^1(\Gamma, \mathbf{C}^4/\mathbf{Z}^4)$ with $v(\gamma_4) = v_0$ for any element v_0 in $\mathbf{C}^4/\mathbf{Z}^4$. Hence the restriction $q_{|L}$ to L of the quotient map $q : Z^1(\Gamma, \mathbf{C}^4/\mathbf{Z}^4) \to H^1(\Gamma, \mathbf{C}^4/\mathbf{Z}^4)$ is surjective. If $(\partial v_0)(\gamma_4) = 0$ for an element v_0 in $\mathbf{C}^4/\mathbf{Z}^4$, then v_0 is in $\mathbf{Z}\frac{1}{3}{}^t(1, 1, 1, 1)$ and hence $(\partial v_0)(\gamma_2) = (\partial v_0)(\gamma_3) = 0$. Therefore, $L \cap B^1(\Gamma, \mathbf{C}^4/\mathbf{Z}^4) = \{0\}$, i.e., $q_{|L}$ is injective. Thus we obtain:

Proposition 34. $H^1(\Gamma, T_N) \simeq \left(\mathbf{C}^t(1, -2, -2, 1) + \frac{1}{2}\mathbf{Z}^4\right)/\mathbf{Z}^4$.

Although Δ does not satisfy the condition of Theorem 24, the barycentric subdivision of Δ with the $\mathbf{Z}$-weight as in Figure 33 does and gives the Γ-equivariant blowing up of Y along all orbits whose closures are compact. Hence the conclusion of Theorem 24 holds.

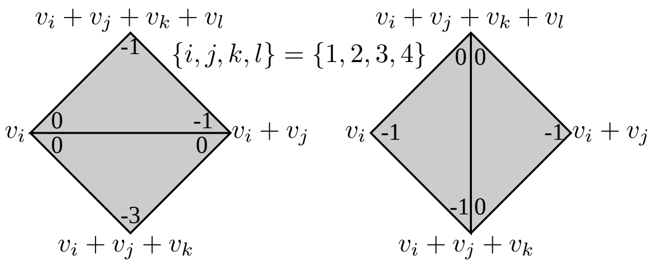


Figure 33

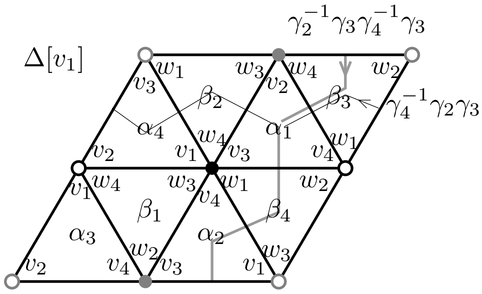


Figure 34

The subgroup of Γ fixing ${}^t(1, 0, 0, 0)$ is a free abelian group generated by $\gamma_4^{-1}\gamma_2\gamma_3$ and $\gamma_2^{-1}\gamma_3\gamma_4^{-1}\gamma_3$ (see Figure 34). Let c be a complex number and let v be the element in $Z^1(\Gamma, \mathbf{C}^4/\mathbf{Z}^4)$ detemined by $v(\gamma_2) = c^t(1, -2, -2, 1)$, $v(\gamma_3) = \gamma_4 v(\gamma_2)$ and $v(\gamma_4) = 0$. Then $v(\gamma_4^{-1}\gamma_2\gamma_3) = c^t(-7, 2, -4, 2)$ and $v(\gamma_2^{-1}\gamma_3\gamma_4^{-1}\gamma_3) = c^t(-8, 4, -5, 1)$. On the other hand, $\mathbf{e}^* := (0, 1, 1, 1)$ is on ∂C^*, $\langle \mathbf{e}^*, {}^t(1, 0, 0, 0) \rangle = 0$. and $\mathbf{e}^*\gamma = \mathbf{e}^*$ for every element γ in $\Gamma_{{}^t(1,0,0,0)}$. Let t be any element in $Z^1(\Gamma, CT_N)$ with $t(\gamma_4) = 1$. Since $\langle \mathbf{e}^*, {}^t(-7, 2, -4, 2) \rangle = \langle \mathbf{e}^*, {}^t(-8, 4, -5, 1) \rangle = 0$, we can define the sum

$$\sum_{[\gamma] \in \Gamma_{{}^t(1,0,0,0)} \backslash \Gamma} t(\gamma)(2\mathbf{e}^*)\mathbf{e}^{2\mathbf{e}^*\gamma},$$

which converges on $U(t)$, because $|t(\gamma)(2\mathbf{e}^*)| = 1$. Moreover, it is $\Gamma(t)$-invariant and gives a fibration on $X(t)$. Also $\sum_{\gamma \in \Gamma} t(\gamma)(x)\mathbf{e}^{x\gamma}$ converges and is $\Gamma(t)$-invariant for any element x in $\mathrm{Int}(C^*) \cap M$. Hence the conclusion of Theorem 24 holds also for $U(t)$, $X(t)$ and $Z(t)$.

References

[1] R. C. Gunning and H. Rossi, Analytic Functions of Several Variables, Prentice-Hall, Englewood cliff, N. J., 1965.

[2] K. Kodaira, On compact analytic surfaces II, Ann. of Math. 77 (1963), 563-626.

[3] Y. Namikawa, Toroidal degeneration of Abelian varieties, II, Math. Ann. 245 (1979).

[4] T. Oda, Convex Bodies and Algebraic Geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge-Band 15, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1987.

[5] H. Tsuchihashi, Higher dimensional analogues of periodic continued fractions and cusp singularities, Tohoku Math. J. 35 (1983), 607-639.

[6] H. Tsuchihashi, Examples of four dimensional cusp singularities, to appear in J. of Math. Soc. Japan.

[7] K. Ueno, Classification Theory of Algebraic Varities and Compact Complex Spaces, Lect. Notes in Math., 439, Springer, Berlin-Heidelberg-New York, 1975.

[8] È. B. Vinberg, Discrete Linear Groups generated by Reflections, Math. USSR Izvestija Vol. 5 (1971), 1083-1119.